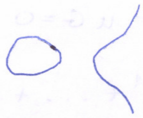
Non-precise def: 1-dim geom object described by polynomial equations.

Def. (for this lecture) $\boxed{\text{Algebraic curve:}}$ $C \subset \mathbb{C}^2$, $C = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$, $f \in \mathbb{C}[x,y]$ nonconstant.

Examples.

f	$C \cap \mathbb{R}^2$	deg
$x+y+1$		1
x^2+y^2-1		2
$(y^2-x^3)^2 - 4x^5y + x^6 - x^7$		7
$y^2 - x^3 + x$		3
x^2+y^2+1		2

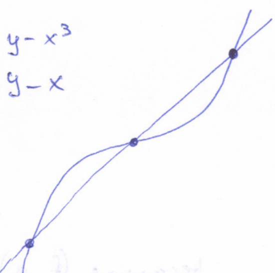
Def. Let $p \in \mathbb{C}[x_1, \dots, x_k]$, then $\boxed{\deg p} := \min\left\{ n \in \mathbb{N} \cup \{-\infty\} \;\middle|\; p = \sum_{i_1,\dots,i_k} a_{i_1,\dots,i_k} x_1^{i_1} \cdots x_k^{i_k} \;\&\; i_1, \dots, i_k \le n \right\}$

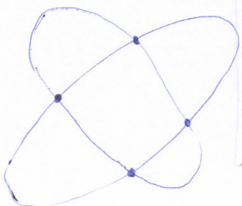
How do we see the degree geometrically? $\longrightarrow$ intersect with a (general) line and count the intersection points

More generally: $C, C' \subset \mathbb{C}^2$ alg curves def'd by $f, g \in \mathbb{C}[x,y]$, $\deg f = m$, $\deg g = n$

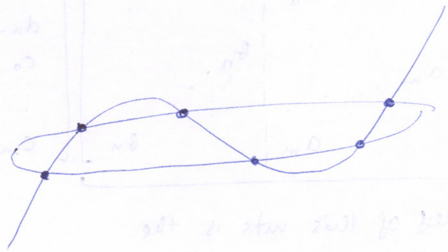
Then what can be said about $C \cap C' \subset \mathbb{C}^2$? Specifically, $\#(C \cap C') = ?$

Ex. $f = y - x^3$
$g = y - x$

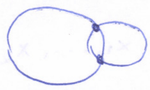




2 ellipses



$\#(C \cap C') = 3 = 3 \cdot 1$ $\qquad$ $\#(C \cap C') = 4 = 2 \cdot 2$ $\qquad$ $\#(C \cap C') = 6 = 3 \cdot 2$ $\Big\} \overset{?}{\Rightarrow} \# = m \cdot n$

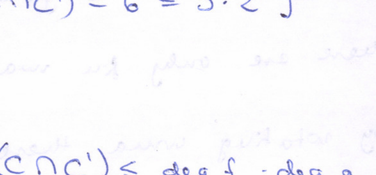
But 2 circles may have 2 intersections: 

Or even 0:  Still, it seems like $\#(C \cap C') \le \deg f \cdot \deg g$

Pathological case: $C = C' \Rightarrow \#(C \cap C') = \infty$

Theorem. (Bézout) $C, C' \subset \mathbb{C}^2$ alg curves as above. If $(f,g) = 1$ then $\#(C \cap C') \le m \cdot n$.

Pf: Idea: scan through the curves with a line, and try to "notice" when we hit an intersection point.



$f(x,y) = a_0(x) y^m + \dots + a_m(x)$, $\qquad g(x,y) = b_0(x) y^n + \dots + b_n(x)$

When do $f(x_0, y)$ and $g(x_0, y) \in \mathbb{C}[y]$ have a common root?

Exercise. $X := \{(x,y,z) \in \mathbb{C}^3 \mid xy = yz = zx = 0\}$ Show that $\nexists f, g : X = \{(x,y,z) \in \mathbb{C}^3 \mid f(x,y,z) = 0 \ \& \ g(x,y,z) = 0\}$

Hint: $\mathbb{P}^2\mathbb{C}$ + Bézout.

For $F, G \in \mathbb{C}[x]$ if $F(a) = G(a) = 0$ for some $a \in \mathbb{C}$ then $F(x) = (x-a)\cdot u(x)$
$G(x) = (x-a)\cdot v(x)$

$\Rightarrow$ $vF - uG = 0$ and $\deg u < \deg F$, $\deg v < \deg G$

The converse holds too. $\longrightarrow$ We can use this to detect common roots of f and g.

$\longrightarrow$ Solve $vF - uG = 0$ for u, v where

$$\underbrace{\begin{aligned} F(x) &= a_0 x^m + \dots + a_m \\ G(x) &= b_0 x^n + \dots + b_n \end{aligned}}_{\text{given}} \qquad \underbrace{\begin{aligned} u(x) &= c_0 x^{m-1} + \dots + c_{m-1} \\ v(x) &= d_0 x^{n-1} + \dots + d_{n-1} \end{aligned}}_{\text{to be found}}$$

$$vF - uG = (a_0 d_0 - b_0 c_0)\cdot x^{m+n-1} + (a_1 d_0 + a_0 d_1 - b_1 c_0 - b_0 c_1)\, x^{m+n-2} + \dots$$

$\rightarrow$ yields $m+n$ linear equations in the variables $c_0, \dots, c_{m-1}, d_0, \dots, d_{n-1}$

$$\begin{array}{c c} & \overbrace{\qquad\qquad\qquad}^{n \text{ columns}} \ \overbrace{\qquad\qquad\qquad}^{m \text{ columns}} \\ m+n \text{ rows} & \underbrace{\begin{bmatrix} a_0 & & & \vdots & b_0 & & \\ a_1 & a_0 & & \vdots & \vdots & \ddots & \\ \vdots & \vdots & \ddots & \vdots & & & b_0 \\ & & a_0 & \vdots & & & \\ a_m & \vdots & & \vdots & b_n & & \vdots \\ & a_m & \ddots & \vdots & & \ddots & \\ & & a_m & \vdots & & & b_n \end{bmatrix}}_{} \end{array} \begin{bmatrix} d_0 \\ \vdots \\ d_{n-1} \\ c_0 \\ \vdots \\ c_m \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}$$

the det of this mtx is the resultant $R_{F,G}$

because $(f,g) = 1$

If $R_{F,G}(x) = 0$ then $\exists y \in \mathbb{C}$: $f(x,y) = g(x,y)$. Since $R_{F,G}$ is a non-constant polynomial in x, there are only fin many $x_1, \dots, x_k$ s.t. $f(x_i, y) = g(x_i, y)$. $\Rightarrow \#(C \cap C') < \infty$

By rotating wma there is at most one intersection point on each vertical line.

Now we estimate $\deg R_{F,G}$, which will give us an upper bound for $\#(C \cap C')$.

Claim. $\deg R_{F,G} \leq mn$

Pf: $R_{F,G}(tx) = \begin{vmatrix} a_0 & & & \\ ta_1 & a_0 & & \dots \\ t^2 a_2 & ta_1 & a_0 & \\ \vdots & \vdots & \ddots & \end{vmatrix} = t^{m\cdot n}\cdot R_{F,G}(x)$ when $a_i(x) = x^i$, $b_j(x) = x^j$

Notice that $\deg a_i \leq i$, $\deg b_j \leq j$ due to degree counting:
$(\deg a_i y^{m-i} = \deg a_i + \deg y^{m-i} \leq m \Rightarrow \deg a_i(x) \leq i)$

We get the exponent $m \cdot n$ as $\left(1+\dots+(m+n-1)\right) - \left(1+\dots+(m-1)\right) - \left(1+\dots+(n-1)\right)$

(matrix operations)

For general $a_i, b_j \to$ only the highest degree matters.

Consider $p_1(x), \dots, p_k(x) \in \mathbb{R}[x]$, and draw their graphs.

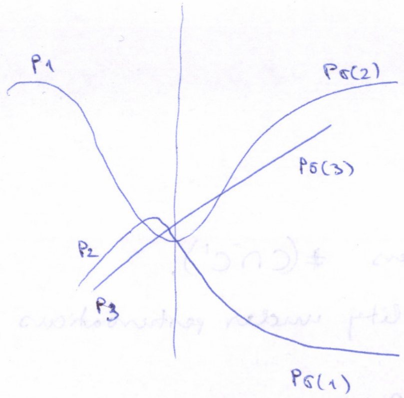
Exercise[4]: For $p_1, \dots, p_k$ pairwise distinct, define a permutation:

For small $x<0$, $p_1(x) < \dots < p_k(x)$ may be assumed.

Then for small $x>0$, $p_{\sigma(1)}(x) < \dots < p_{\sigma(k)}(x)$ for some $\sigma \in S_k$

In this context, small means small in absolute value; more precisely:

$\exists a>0$ s.t. $|x|<a$ and $\forall y \in \mathbb{R}, |y|<a$: $p_i(y) \neq p_j(y)$ $\forall i \neq j$.



Show:

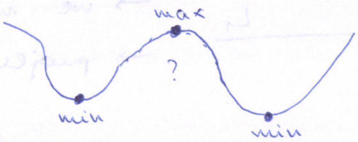
a) For $k \leq 3$, every permutation can be realised.

b) For $k=4$, there is one that cannot be realised (and find such a permutation).

Exercise[5a]: $p \in \mathbb{R}[x]$, $p(x) > 0$ $\forall x \in \mathbb{R}$ $\Rightarrow 2 \mid \deg p$ & p has a global minimum. (known)

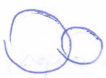
To generalise, consider $p(x,y) = x^2y^2 + 2xy + x^2 + 1 \in \mathbb{R}[x,y]$. Show $p(x,y) > 0$ $\forall (x,y) \in \mathbb{R}^2$ but has no minimum.

Exercise[5b]: Reminder: if a polynomial in 1 variable has 2 local minima then it has a local maximum too.



Show that $p(x,y) = (x^2y - x - 1)^2 + (x^2-1)^2$ has 2 global minima, but no other critical points.

Exercise[2]: Where are the missing pts?



Recall Exercise 4 from last time.

Hint. This cannot be realised: $(1,2,3,4) \mapsto (2,4,1,3)$.

Any $f \in \mathbb{R}[x]$ can be written as $f(x) = a_m x^m + \dots + a_d x^d$, $a_m, a_d \neq 0$.

$\boxed{m_0(f)} := m$ $\boxed{\text{multiplicity}}$ of f at 0 / valuation of f

$\boxed{\deg(f)} := d$ $\boxed{\text{degree}}$ of f

Note that for $|x| \gg 0$, $f(x) \sim a_d x^d$,

for $|x| \ll 1$, $f(x) \sim a_m x^m$.

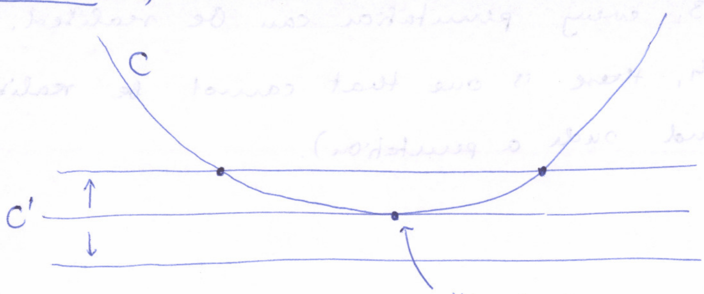
If $2 \mid m_0(f)$, f stays on the same side of the x-axis,

if $2 \nmid m_0(f)$, f changes sides.

If $m_0(f) > m_0(g)$ then for $|x| \ll 1$, $|f(x)| < |g(x)|$

Goal. Get equality in Bézout.

Problems. 1)



$C = \{y - x^2 = 0\}$

$C' = \{y = 0\}$

Wiggling C' alters $\#(C \cap C')$.

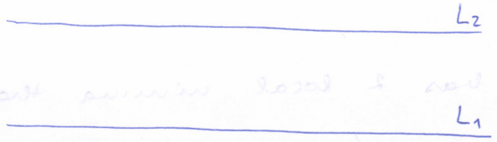
We want stability under perturbations.

Want this to have multiplicity 2:

$$\left.\begin{array}{l} y - x^2 = 0 \\ y = 0 \end{array}\right\} \Rightarrow x^{2} = 0$$

Solution: define intersection multiplicities.

2)



should have 1 intersection.

$\to$ we wish for points at infinity

$\to$ projective space.

Projective spaces

$\boxed{\mathbb{P}^n(\mathbb{C})} := \mathbb{C}^{n+1} \setminus \{0\} / \mathbb{C}^\times$ $\boxed{\mathbb{P}^n(\mathbb{R})} := \mathbb{R}^{n+1} \setminus \{0\} / \mathbb{R}$

Ex. $\mathbb{P}^1(\mathbb{R}) \simeq S^1$, $\mathbb{P}^3(\mathbb{R}) \simeq SO(3)$ as groups, $\mathbb{P}^1(\mathbb{C}) \cong S^2$ diffeomorphism (exercise)

$$\varphi_i : \mathbb{C}^n \hookrightarrow \mathbb{P}^n(\mathbb{C})$$

$$(x_1, \dots, x_n) \mapsto [x_1 : \dots : x_{i-1} : \overset{i}{1} : x_i : \dots : x_n]$$

$$\mathbb{P}^n(\mathbb{C}) \setminus \varphi_i(\mathbb{C}^n) = \{[x_0 : \dots : \overset{i}{0} : \dots : x_n] \mid x_1, \dots, x_n \in \mathbb{C} \text{ not all } 0\} \cong \mathbb{P}^{n-1}(\mathbb{C})$$

Def. $\boxed{\text{Projective algebraic curve}}$: $C \subset \mathbb{P}^2(\mathbb{C})$, $C = \{[x:y:z] \in \mathbb{P}^2(\mathbb{C}) \mid F(x,y,z) = 0\}$

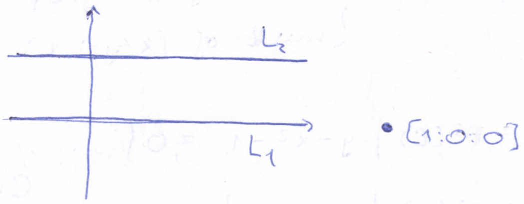
where $F \in \mathbb{C}[x,y,z]$ is nonconstant & homogeneous.

Homogeneous polynomials, homogenisation.

$$\{C \subset \mathbb{C}^2\} \longleftrightarrow \{\overline{C} \subset \mathbb{P}^2(\mathbb{C})\}$$

$$C = \{f = 0\} \longleftrightarrow \{F = 0\} = \overline{C}$$ where F is the homogenisation

Ex. $L_1 = \{(x,0)\}$, $L_2 = \{(x,1)\} \subset \mathbb{C}^2$



$f_1(x,y) = y$ $\qquad$ $f_2(x,y) = y-1$

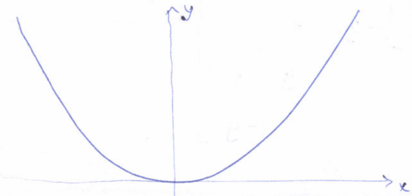
$\downarrow$ homogenise $\downarrow$

$F_1(x,y,z) = y$ $\qquad$ $F_2(x,y,z) = y - z$ $\longrightarrow$ intersections in $\mathbb{P}^2(\mathbb{C})$:

$$\left.\begin{matrix} y=0 \\ y-z=0 \end{matrix}\right\} \Rightarrow y = z = 0 \Rightarrow x \neq 0 \Rightarrow \overset{\infty}{[\underbrace{1:0}:0]}$$

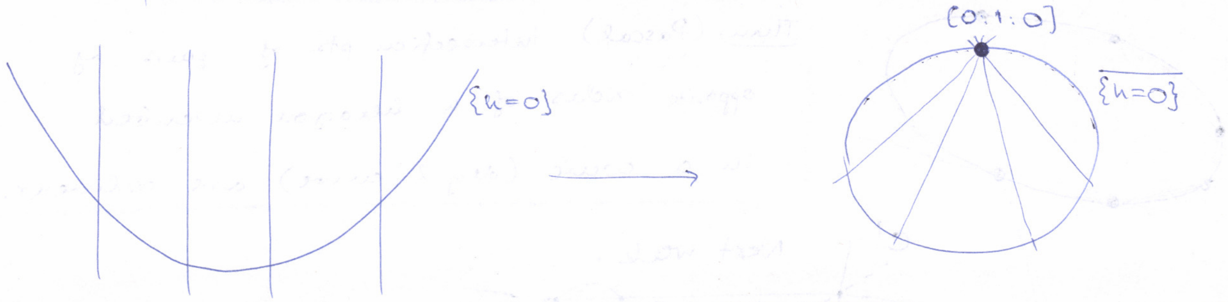
the point is in the direction of the x-axis

Ex. $h(x,y) = y - x^2 \longrightarrow yz - x^2$



point(s) at infinity:

$[0:1:0] \in \overline{\{h=0\}} \subset \mathbb{P}^2\mathbb{C}$



Theorem. (Bézout in $\mathbb{P}^2$) $C, C' \subset \mathbb{P}^2\mathbb{C}$ algebraic curves without common components.

$$\Rightarrow \#_{mult}(C \cap C') := \sum_{P \in C \cap C'} i_P(C, C') = \deg C \cdot \deg C'$$

where $i_P(C, C') \in \mathbb{Z}$ is the intersection multiplicity.

Do the same as before: $F(x,y,z) = \sum_{i=0}^{m} a_i(x,y) z^{m-i}$ $\qquad$ $a_i, b_i \in \mathbb{C}[x,y]$ homog of deg i

$$G(x,y,z) = \sum_{i=0}^{n} b_i(x,y) z^{n-i}$$

$$\det \begin{bmatrix} a_0(x,y) & & 0 & -b_0(x,y) & & 0 \\ \vdots & \ddots & & \vdots & \ddots & -b_0(x,y) \\ a_m(x,y) & & a_0(x,y) & \vdots & & \vdots \\ & \ddots & \vdots & -b_n(x,y) & & \vdots \\ 0 & & a_m(x,y) & 0 & \ddots & -b_n(x,y) \end{bmatrix} =: R_{F,G}(x,y) \in \mathbb{C}[x,y]$$

homog of deg $m \cdot n$

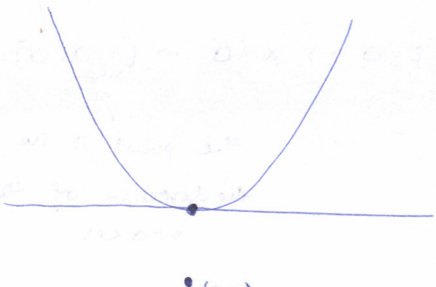
<u>Def.</u> For $P \in C \cap C'$, let $\boxed{i_P(C,C')}$ be the multiplicity of the zero $[x:y]$ of $R_{F,G}(x,y)$.

Alternatively: $i_P(C,C') = \begin{cases} \text{mult. of } [1 : y/x] \text{ in } R_{F,G}(1,Y) & \text{if } x \neq 0, \\ \text{mult. of } [x/y : 1] \text{ in } R_{F,G}(X,1) & \text{if } y \neq 0 \end{cases}$

<u>Ex.</u> $C = \{[x:y:z] \in \mathbb{P}^2(\mathbb{C}) \mid yz - x^2 - z^2 = 0\}$

$C' = \{[x:y:z] \in \mathbb{P}^2(\mathbb{C}) \mid y - z = 0\}$

$C \cap C' = \{P = [0:1:1]\}$

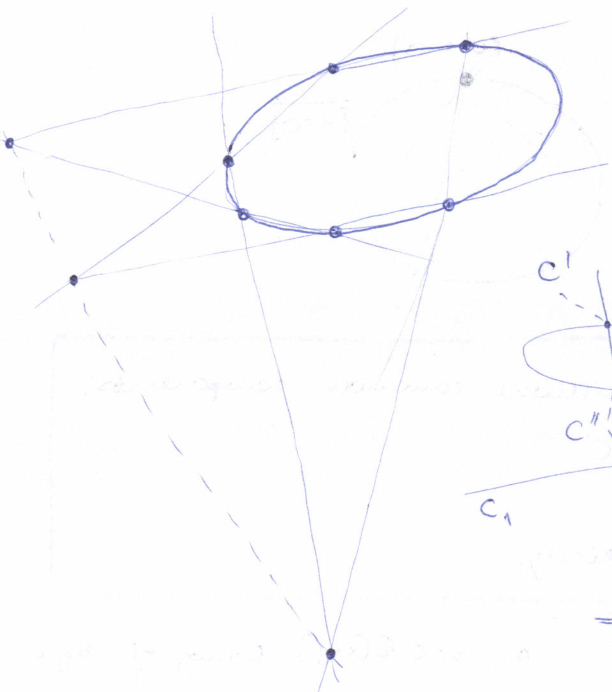


$F(x,y,z) = (-1) z^2 + y \cdot z^1 - x^2 \cdot z^0$

$G(x,y,z) = z^1 - y \cdot z^0$

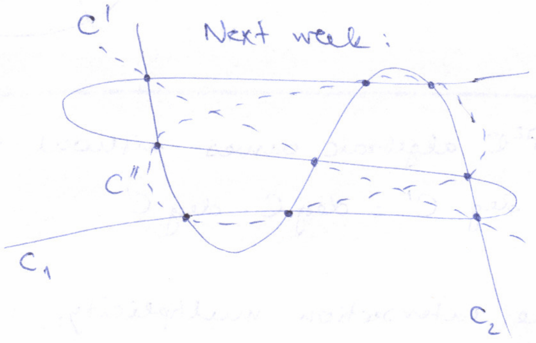
$$R_{F,G}(x,y) = \det \begin{bmatrix} -1 & 1 & \\ y & -y & 1 \\ -x^2 & & -y \end{bmatrix} = -x^2$$

$\Rightarrow i_P(C,C') = 2$



<u>Thm.</u> (Pascal) Intersection pts of pairs of opposite sides of a hexagon inscribed in a conic (deg 2 curve) are colinear.

Next week:

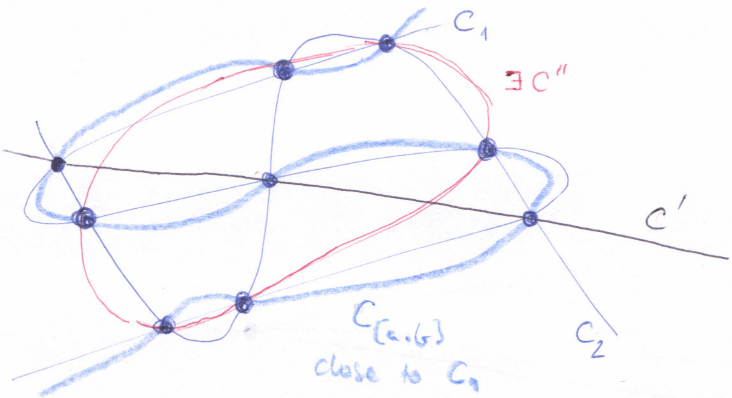


$\deg C_1 = \deg C_2 = d$

$C_1 \cap C_2 = \{d^2 \text{ points}\}$

$\deg C' = k$ irreducible, contains $k \cdot d$ of these pts

$\Rightarrow \exists C''$, $\deg C'' = d - k$ going through the rest.



$\deg C_1 = \deg C_2 = d$

$\deg C' = k$ irreducible

$\deg C'' = d-k$

Lemma. $C_1, C_2 \subset \mathbb{P}^2(\mathbb{C})$ curves of degree d, $C_1 \cap C_2 = \{ d^2 \text{ points}\}$. Assume there is an irreducible curve C' of degree $k < d$ passing through $k \cdot d$ pts of $C_1 \cap C_2$. Then there exists a curve C'' of degree $d-k$ through the remaining $(d-k)\cdot d$ points.

Pf: $C_i = \{F_i = 0\}$, $F_i \in \mathbb{C}[x,y,z]$ homog of deg d

For $[a:b] \in \mathbb{P}^1(\mathbb{C})$ consider the curve $C_{[a:b]} := \{ aF_1 + bF_2 = 0\}$

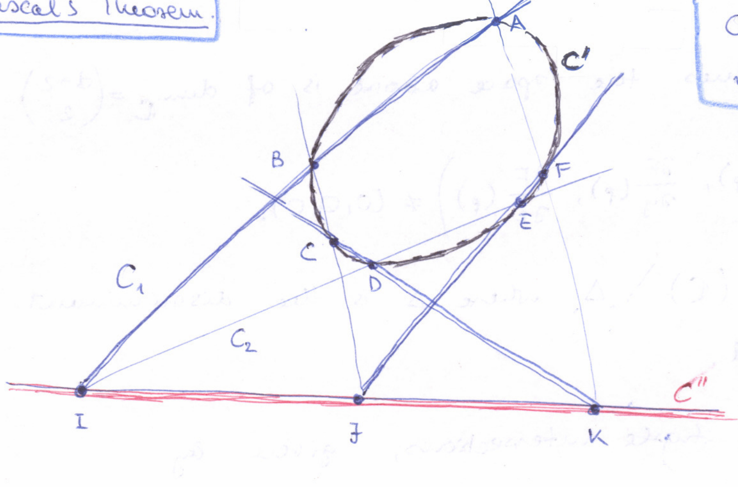
Then $\deg C_{[a:b]} = d$ and $\forall [a:b] \in \mathbb{P}^1\mathbb{C}$: $(C_1 \cap C_2) \subset C_{[a:b]}$. Think of this as an algebraic version of an isotopy between curves.

Let $p \in C' \setminus (C_1 \cap C_2)$, $a := F_2(p)$, $b := F_1(p)$ $\Rightarrow$ $[a:b] \in \mathbb{P}^1(\mathbb{C})$,

$C' \cap C_{[a:b]}$ contains at least $\underbrace{k \cdot d}_{C' \cap C_1 \cap C_2 \not\ni p} + \underbrace{1}_{p}$ points $\Rightarrow$ C' and $C_{[a:b]}$ share a common component by Bézout.

Since C' is irreducible: $C_{[a:b]} = C' \cup C''$ for some curve C'' of degree $d-k$.

Pascal's Theorem



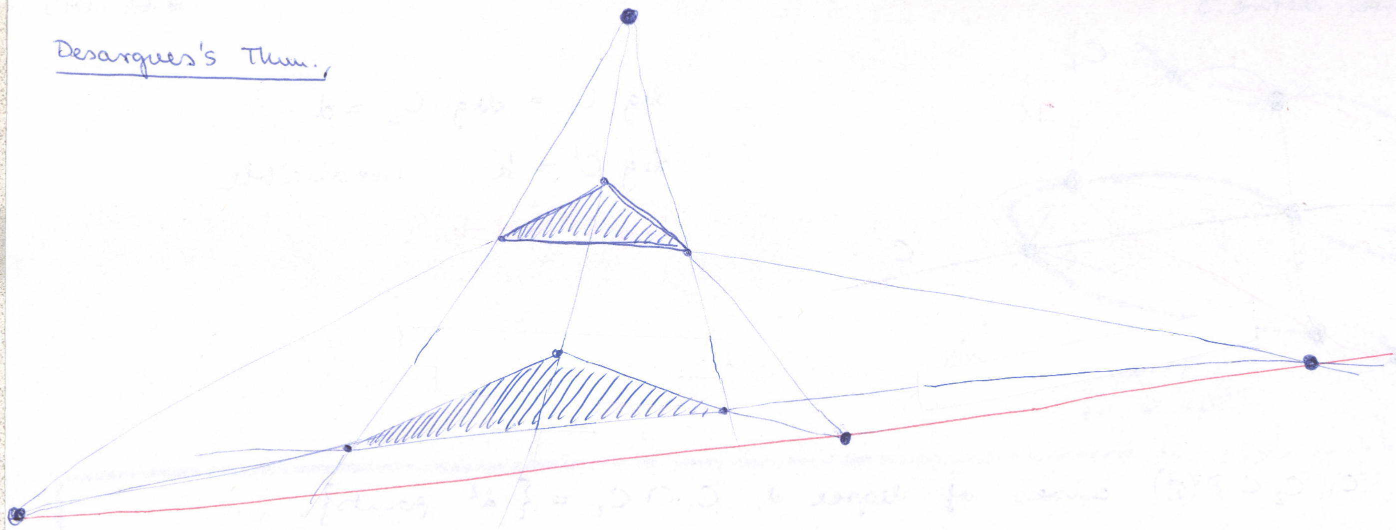
Claim: I, J, K are colinear, where C' is a smooth conic in $\mathbb{P}^2\mathbb{C}$.

Pf: Use the Lemma with

$C_1 := \overline{AB} \cup \overline{CD} \cup \overline{EF}$

$C_2 := \overline{BC} \cup \overline{DE} \cup \overline{AF}$

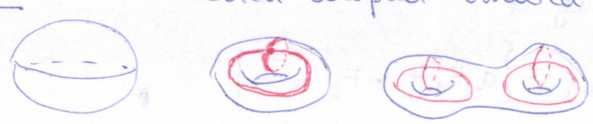
Desargues's Thm.



Go one dimension higher, look at the planes of the triangles.

Smooth curves

Recall: connected compact oriented surfaces are classified by their genera.



The genus tells us how many "broken glasses"  we can embed

(This is how the classification result is proven.)

Fix some $d \in \mathbb{N}$ and consider $\boxed{\mathcal{C}_d} := \{ C = \{F(x,y,z) = 0\} \subset \mathbb{P}^2(\mathbb{C}) \mid F \in \mathbb{C}[x,y,z] \text{ homog of deg } d \}$

$F(x,y,z) = \sum_{i+j+k=d} a_{ijk} x^i y^j z^k$ for some $a_{ijk} \in \mathbb{C}$ not all zero.

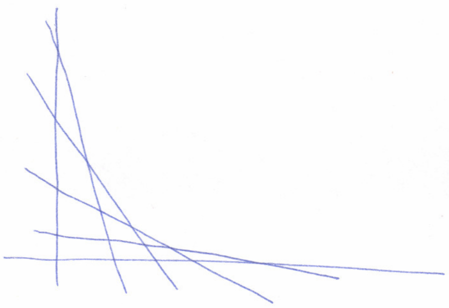
There are $\binom{d+2}{2}$ terms in this Σ, thus the space above is of $\dim_{\mathbb{C}} = \binom{d+2}{2}$

Def $C = \{F = 0\}$ is $\boxed{\text{smooth}}$ if $\forall p \in C: \left(\frac{\partial F}{\partial x}(p), \frac{\partial F}{\partial y}(p), \frac{\partial F}{\partial z}(p)\right) \neq (0,0,0)$.

$\{\text{smooth curves of degree } d\} \cong \mathbb{P}^{\binom{d+2}{2}-1}(\mathbb{C}) \setminus \Delta$ where Δ is the discriminant.

Note that this space is path-connected.

Let $L_1, \dots, L_d \subset \mathbb{P}^2\mathbb{C}$ be lines without triple intersections, given by $l_1, \dots, l_d \in \mathbb{C}[x,y]$ linear forms.

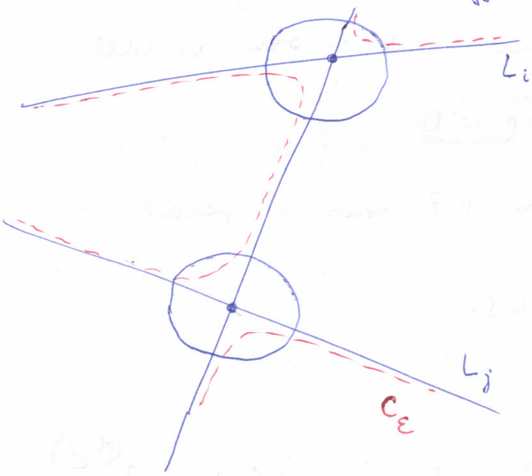


Consider $C_0 := \{ l_1 \cdot l_2 \cdot \ldots \cdot l_d = 0\}$

and perturb it slightly: $C_\varepsilon := \{ l_1 \cdots l_d - \varepsilon l_0^d = 0 \}$

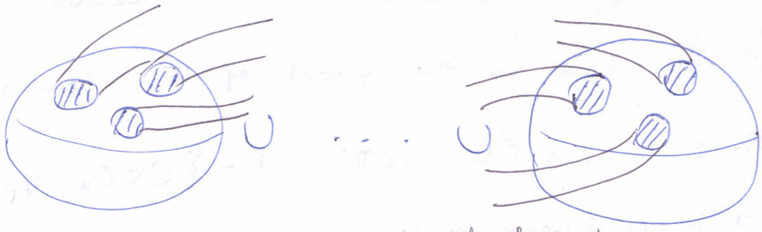
where l_0 is chosen s.t. C_ε becomes smooth for small $\varepsilon > 0$.

Such an l_0 exists: it suffices for it not to vanish at any of the intersections.

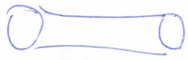


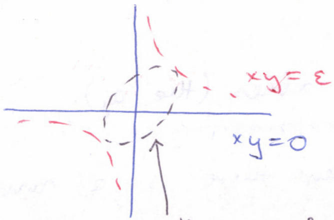
$\mathbb{P}^1\mathbb{C} \cong S^2$

This is $S^2 \setminus (d-1 \text{ discs})$ d times:



$\{xy=\varepsilon\} \cap B(0,R) \cong$ cylinder





this circle is visible only over $\mathbb{C}$,
red and black together give a cylinder

$$\chi(C_\varepsilon) = d\cdot\chi\left(S^2\setminus(d-1 \text{ discs})\right) + \ldots\cdot\underbrace{\chi(\text{tube})}_{0} - \underbrace{\ldots\cdot\underbrace{\chi(S^1)}_{0}}_{\text{what we counted twice}}$$

$$= d\cdot(2-d+1) = -d^2+3d$$

$$g(C_\varepsilon) = \frac{(d-1)(d-2)}{2}$$

Notice that these depend on $d=\deg F$ only.

23.04.2019

<u>Recall</u> the correspondence $C_d \longleftrightarrow \mathbb{P}^{\binom{d+2}{2}-1}\mathbb{C}$

$$C=\{F=0\} \longleftrightarrow [a_1 : \ldots : a_{\binom{d+2}{2}}]$$

where $F = \sum_{i=1}^{\binom{d+2}{2}} a_i M_i$ where M_i is a monomial

Notice that this is rather curious: e.g. groups don't form a group, but here we have that curves form a projective space.

<u>Def.</u> A $\boxed{\text{linear system of degree } d \text{ curves}}$ is a subset $V \subseteq C_d$ whose image in $\mathbb{P}^{\binom{d+2}{2}-1}\mathbb{C}$ is given by linear equations.

Special cases: $C_1 = \{\text{lines}\} \longleftrightarrow \mathbb{P}^2\mathbb{C}$ this is how $\mathbb{P}^2$ was def'd

$C_2 = \{\text{quadrics}\} \longleftrightarrow \mathbb{P}^5\mathbb{C}$

$C_3 = \{\text{cubics}\} \longleftrightarrow \mathbb{P}^9\mathbb{C}$

Example. Fix a conic $C \subset \mathbb{P}^2\mathbb{C}$, and let $V := \{T_pC \mid p \in C\}$ be the set of tangent lines. $V \subset \mathcal{C}_1 \cong \mathbb{P}^2\mathbb{C}$. Exc.: show that V is a conic as well.

Def. A linear system of dimension 1 is called a $\boxed{\text{pencil}}$.

E.g. what we had in the proof of the Lemma on p.7 was a pencil.

Example. Fix $p_1, \dots, p_n \in \mathbb{P}^2\mathbb{C}$, $d \in \mathbb{N}$, $H := \{C \in \mathcal{C}_d \mid \forall p_i \in C\}$.

Then H is a linear system.

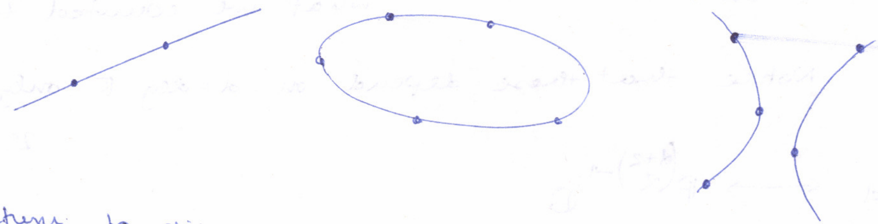
$F = \sum_{j=1}^{\binom{d+2}{2}} a_j M_j$, we get a system of equations given by $\{F(p_j) = 0\}_{j=1}^{\binom{d+2}{2}}$

We have n equations and $\binom{d+2}{2}$ variables (the a_j).

Lemma. Let $n, d \in \mathbb{N}$ s.t. $n < \binom{d+2}{2}$. Then there is a curve of degree d through any set of n points. (Just linear algebra.) $\square$

Ex. $2 < \binom{1+2}{2} \longrightarrow$ there is a line through any 2 points

$5 < \binom{2+2}{2} \longrightarrow$ there is a quadric through any 5 points



We return to discussing smooth curves.

Recall $\{C \in \mathcal{C}_d \mid C \text{ smooth}\} \hookrightarrow \mathbb{P}^{\binom{d+2}{2}-1}\mathbb{C}$, more precisely $\{C \in \mathcal{C}_d \text{ smooth}\} = \mathbb{P}^{\binom{d+2}{2}-1}\mathbb{C} \setminus \Delta$ where Δ is a hypersurface called the discriminant.

This is a purely theoretical result, in practice it is not at all convenient to do computations with Δ. The important thing is that $\mathbb{P}^{\binom{d+2}{2}-1}\mathbb{C} \setminus \Delta$ is path-connected.

Thm. $C \in \mathcal{C}_d$ smooth $\Rightarrow$ C is a compact Riemann surface of genus

$$g(C) = \frac{(d-1)(d-2)}{2}$$

Rmk. There are gaps:

d	1	2	3	4	5	...
g	0	0	1	3	6	...

In particular, there are no curves of genus 2.

There is a converse to this Thm:

Thm. (Chow, 1949) $S \subset \mathbb{P}^2\mathbb{C}$ compact Riemann surface (i.e. locally given by analytic equations). Then S is an algebraic curve (i.e. globally given by a polynomial equation).

This is truly remarkable: we get that S itself is an algebraic curve, not just isotopic (or sth like that) to one.

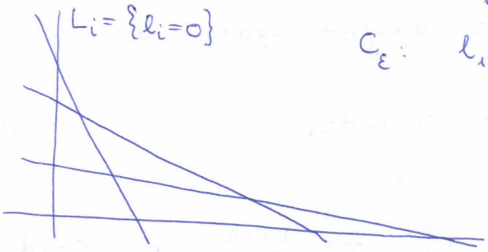
Exc. $C := \{ y = p(x)^2 \} \subseteq \mathbb{C}^2$, $p(x) = (x-a_1)\cdots(x-a_{2g+1})$, $a_i \in \mathbb{C}$ distinct

$\Rightarrow C$ is a non-cpt smooth surface of genus g.

In particular, we can realise all the genera. Why is this not in contradiction with the Rmk. above? (What happens when we projectivise?)

Thm. (degree-genus formula) $C \in \mathcal{C}_d$ smooth $\Rightarrow g(C) = \dfrac{(d-1)(d-2)}{2}$

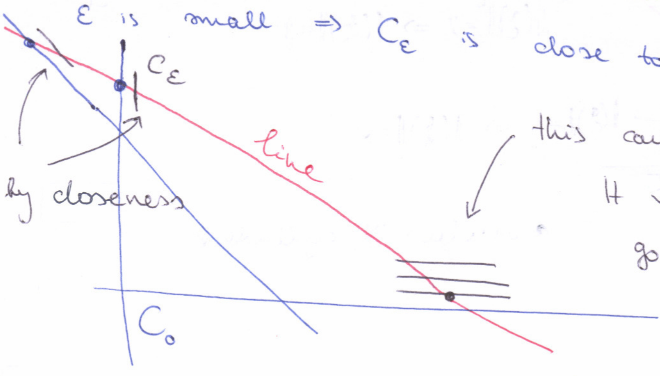
PF SKETCH: Consider $C_0: \ l_1 l_2 \cdots l_d = 0$ where l_i are generic linear forms.



$C_\varepsilon: \ l_1 l_2 \cdots l_d = \varepsilon \cdot l_0^d$ for $\varepsilon > 0$ small s.t. C_ε is smooth of degree d.

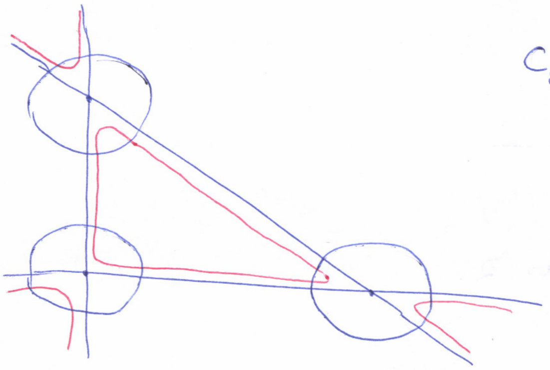
Idea: Choose a path in $\mathbb{P}^{\binom{d+2}{2}-1}\mathbb{C} \setminus \Delta$ connecting C and C_ε $\Rightarrow g(C) = g(C_\varepsilon)$.

ε is small $\Rightarrow C_\varepsilon$ is close to C_0.

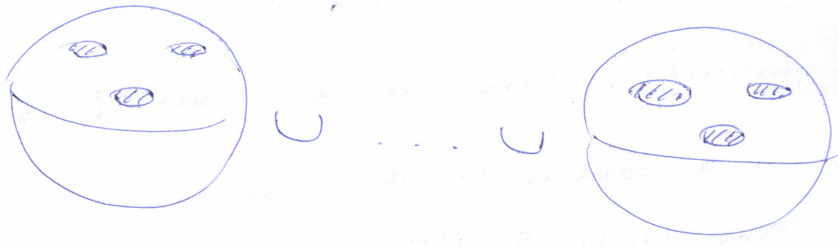


this cannot happen: there can't be multiple layers. It would contradict Bézout: have a line go through them and count the intersections.

What happens near intersection points?



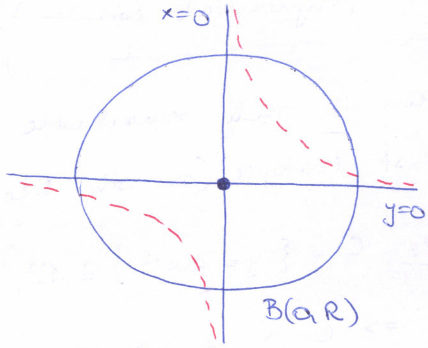
$$C_\varepsilon \setminus \bigcup_{p \in L_i \cap L_j} B(p, R) = \coprod \left(\mathbb{P}^1\mathbb{C} \setminus \{(d-1) \text{ discs}\} \right)$$



d spheres with d-1 discs removed from each

Near an intersection point $p \in L_i \cap L_j$ we may choose coordinates $\mathbb{C}^2 \hookrightarrow \mathbb{P}^2\mathbb{C}$ s.t. $C_\varepsilon \cap B(p, R)$ is described by

$$\{ (x, y) \in \mathbb{C}^2 \mid xy = \varepsilon,\ |x|^2 + |y|^2 \leq R \}$$



$$4xy = (x+y)^2 - (x-y)^2$$

Change of coordinates: $u := x + y$, $iv := x - y$,

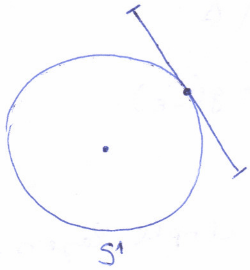
$xy = \varepsilon$ becomes $u^2 + v^2 = 4\varepsilon$. Write $u = a + ib$, $v = c + id$, $a, b, c, d \in \mathbb{R}$

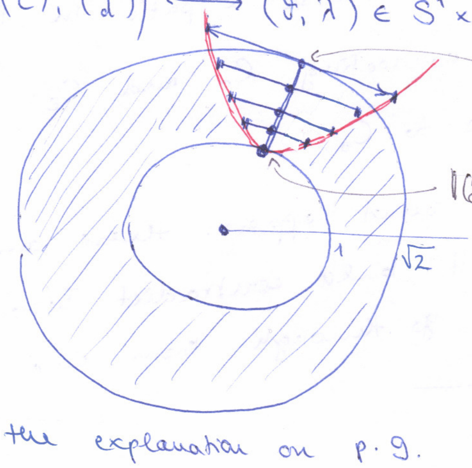
$$\Rightarrow \underbrace{a^2 + c^2}_{\left\| \binom{a}{c} \right\|^2} \underbrace{- b^2 - d^2}_{- \left\| \binom{b}{d} \right\|^2} + 2i \underbrace{(ab + cd)}_{= 0 \text{ as } \binom{a}{c} \perp \binom{b}{d}} = 4\varepsilon$$

Rescale coordinates: $2 \geq \left\| \binom{a}{c} \right\|^2 = \left\| \binom{b}{d} \right\|^2 + 1 \geq 1$

$\Rightarrow \binom{a}{c} \neq \binom{0}{0}$ and $\left\| \binom{b}{d} \right\|^2 \in [0, 1]$. $\longrightarrow$ This is the tangent space to the circle with vectors of bounded length $\longrightarrow$ cylinder.

$$(x, y) \longleftrightarrow (u, v) \longleftrightarrow \left(\binom{a}{c}, \binom{b}{d} \right) \longleftrightarrow (\vartheta, \lambda) \in S^1 \times [-1, 1], \quad \vartheta := \frac{a + ib}{|a + ib|}, \quad \lambda := \frac{c + id}{i\vartheta}$$





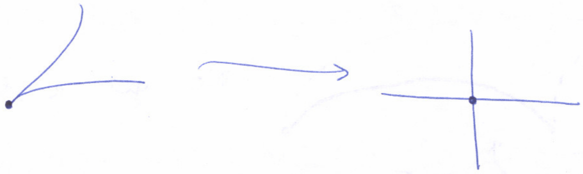
$\left\| \binom{a}{c} \right\|^2 = 2 \Rightarrow \left\| \binom{b}{d} \right\| = 1$

$\left\| \binom{a}{c} \right\| = 1 \Rightarrow \left\| \binom{b}{d} \right\| = 0$

annulus $\cong$ cylinder

This sufficiently amends the explanation on p. 9. □

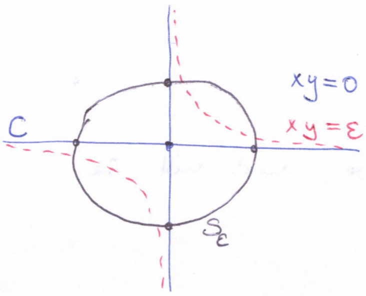
Singularities

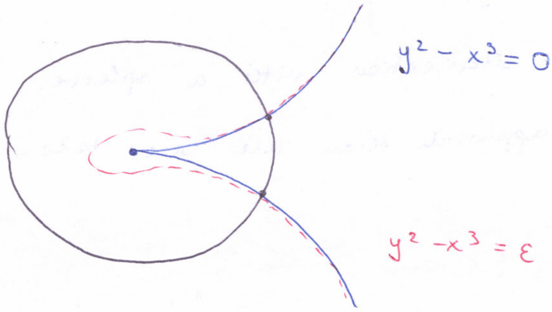


<u>Def.</u> $C = \{[x:y:z] \in \mathbb{P}^2\mathbb{C} \mid F(x,y,z)=0\}$. $p \in C$ is a $\boxed{\text{singular point}}$ if
$$\frac{\partial F}{\partial x}(p) = \frac{\partial F}{\partial y}(p) = \frac{\partial F}{\partial z}(p) = 0.$$

$p \in C$ is a $\boxed{\text{smooth point}}$ if at least one of these partial derivatives doesn't vanish (and then the implicit function theorem can be applied).

30.04.2019





Looking at <u>level sets</u> for small ε, $f=\varepsilon$ is close to $f=0$ except at the singular pt, where it is relatively far away.

Intersect with a sphere and study the intersections.

$S_\varepsilon := \{(x,y) \in \mathbb{C}^2 \mid |x|^2+|y|^2 = \varepsilon^2\}$ 3-dimensional sphere

$C = \{(x,y) \in \mathbb{C}^2 \mid f(x,y)=0\} \ni (0,0)$ (translate so that this holds)

Study $S_\varepsilon \cap C$.

<u>Ex.</u> $f(x,y) = x^2 - y^3$.

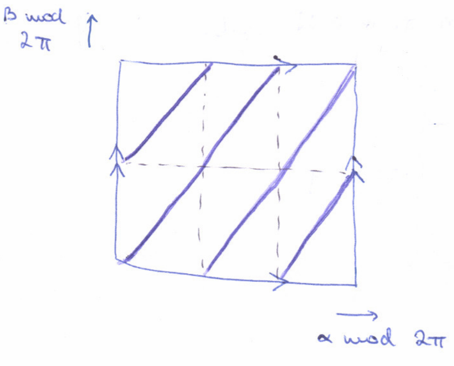
Write $x = re^{i\alpha}$, $y = se^{i\beta}$ polar coordinates, $r,s>0$, $\alpha,\beta \in \mathbb{R}$

$(x,y) \in S_\varepsilon \cap C \iff s^2 e^{2i\beta} = r^3 e^{3i\alpha}$ & $s^2+r^2 = \varepsilon^2$

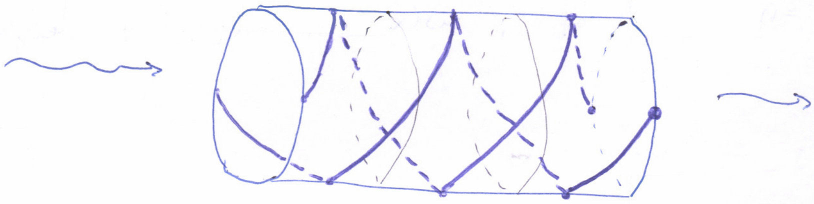
$\iff s^2 = r^3$ & $2\beta - 3\alpha \in 2\pi\mathbb{Z}$ & $\underbrace{r^3+r^2}_{}=\varepsilon^2$

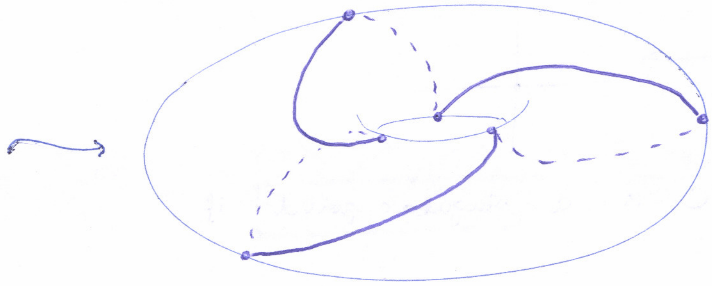
strictly mon increasing
$\Rightarrow$ has unique solution r

$\Rightarrow$ get unique $s = r^{3/2}$ $\Rightarrow$ $(x,y) \in \underbrace{rS^1 \times sS^1}_{\subset S^3_\varepsilon \subset \mathbb{C}^2}$ This is a torus.



$\beta = \frac{3}{2}\alpha + n\pi$
$n \in \mathbb{Z}$





We get a trefoil knot.

(For the time being, we don't care about how these are oriented, positivity, and negativity.)

Now we take a different approach: parametrise the curve:

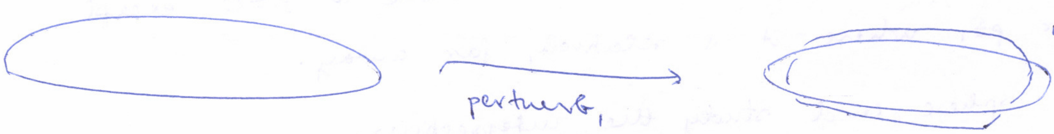
$$\mathbb{C} \longrightarrow C \subset \mathbb{C}^2$$
$$t \longmapsto (t^2, t^3) = (x, y)$$

For $|t|$ fixed, we get the intersection with a sphere.

This is a more general approach than the one taken above, and will be generalised later.

<u>Ex.</u> $f = y^2$

$(x, y) \in S_\varepsilon \cap C \iff |x| = \varepsilon,\ y = 0$



<u>Ex.</u> $f = (\underbrace{y^2 - x^3}_{y_1})^2 - x^7 = y_1^2 - x^7$

We start with this because this term has minimal degree. The x^7 in comparison is much smaller.

$\forall x \neq 0$ there are exactly 2 y_1's close to $y_1 = 0$:

$y_1 = x^{7/2}$, x small, $|x| \approx \varepsilon \Rightarrow |y_1| \approx \varepsilon^{7/2} \ll \varepsilon$

This motivates the following procedure.

<u>Def.</u> A polynomial $f \in \mathbb{C}[x, y]$ is $\boxed{\text{weighted homogeneous}}$ if $\exists n, a, b \in \mathbb{N}$ s.t.

$$f(t^a x, t^b y) = t^n f(x, y) \quad \forall t \in \mathbb{C} \quad \forall (x, y) \in \mathbb{C}^2$$

$w(f) := n$ weight of f, $w(x) := a$, $w(y) := b$ weights of x resp. y

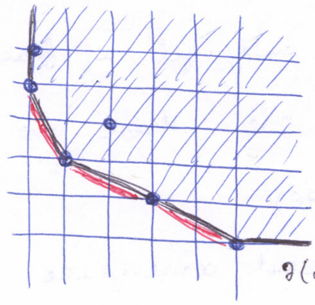
Ex. $f(x,y) = y^2 - x^3 \longrightarrow w(x)=2,\ w(y)=3,\ w(f)=6$ weighted homogeneous

$x^i y^j$ is also weighted homogeneous, $w(x)=j,\ w(y)=i,\ w(x^iy^j)=2ij$

Def. $f = \sum_{ij} a_{ij}\, x^i y^j \in \mathbb{C}[x,y]$ has support $\boxed{\operatorname{supp} f} = \{ (i,j) \in \mathbb{N}^2 \mid a_{ij} \neq 0 \}$

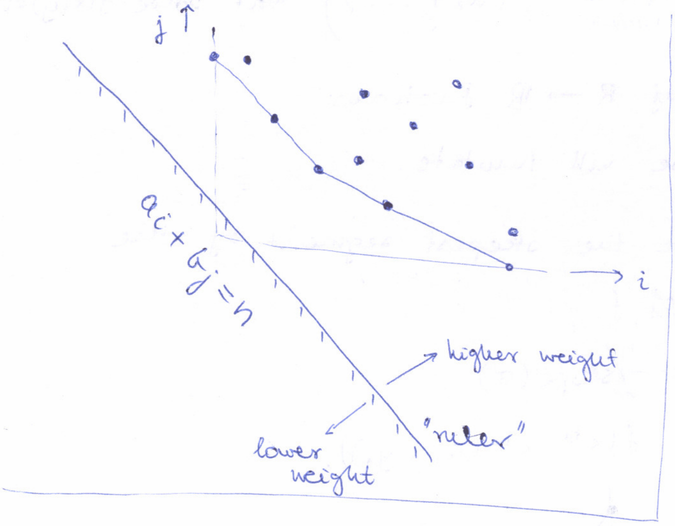
Note that this is of course a notion totally different from that of a support of a function.

Ex. $f = 7y^5 - iy^4 + xy^2 - 3x^2y^3 + x^3y + x^5$



$$\partial\left(\text{convex hull}\left(\bigcup_{p\in \operatorname{supp} f} p + \mathbb{R}^2_{>0}\right)\right)$$ is a polygon with 2 infinite rays

The $\boxed{\text{Newton polygon}}$ of f consists of the finite segments.



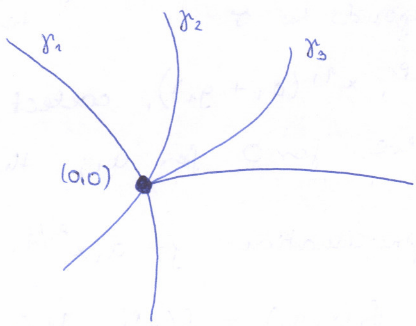
Theorem (Puiseux). Let $C \subset \mathbb{C}^2$ be an algebraic curve. Assume that $(0,0) \in C$, $\{x=0\} \not\subset C$. Then near $(0,0)$ C is the union of finitely many branches $\gamma_1, \dots, \gamma_n \subset \mathbb{C}^2$ where each branch has an injective parametrisation of the form $t \mapsto (t^m, g(t))$ where $m \in \mathbb{N}$, g a holomorphic function (i.e. a power series in t).
Moreover, the branches intersect at $(0,0)$ only.

Equivalently, put $x := t^m$, get. $x \mapsto (x, g(x^{1/m}))$. Such a power series with fractional powers is called a $\boxed{\text{Puiseux series}}$.

Rewrite $g(x^{1/m})$: a $\boxed{\text{Newton series}}$ is

$$x^{\frac{q_1}{p_1}}\left(a_1 + x^{\frac{q_2}{p_1p_2}}\left(a_2 + x^{\frac{q_2}{p_1p_2p_3}}\left(\dots\right)\right)\right)$$

where $\forall (p_i, q_i)$ is a $\boxed{\text{Newton pair}}$, i.e. these are coprime positive integers.



$$b_1 x^{\frac{m_1}{n_1}} + b_2 x^{\frac{m_2}{n_1n_2}} + b_3 x^{\frac{m_3}{n_1n_2n_3}}$$ where $\forall (m_i, n_i)$ is a $\boxed{\text{Puiseux pair}}$, i.e coprime

$p_i := n_i$, $q_1 := m_1$, $q_i := m_i - m_{i-1} n_i$ gives a bijection between these

Goal for today: "solve" $f(x,y)=0$ near $(0,0)$; $f(0,0)=0$.

- If $\frac{\partial f}{\partial y}(0,0) \neq 0$ $\longrightarrow$ the implicit function thm does the job.

  The implicit function thm gives a holomorphic output if the input is holomorphic $\Rightarrow$ we get $y = \sum_{i \geq 0} a_i x^i$ power series

- If $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$, Puiseux's Thm does the job.

  Concession: we only get fractional power series $y = \sum_{i \geq 0} a_i x^{i/N}$, and not only one: there are multiple branches.

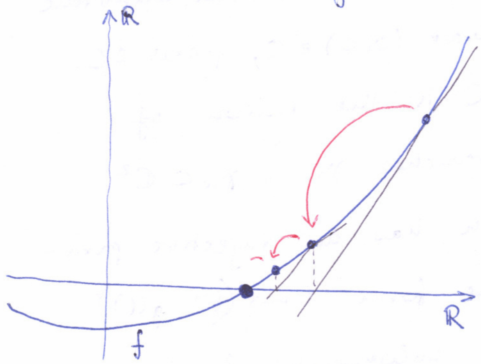
Newton's Algorithm. (1676)

We won't worry about convergence (Newton for sure didn't); perhaps this will be discussed in an upcoming lecture.

Input: $f \in \mathbb{C}[x,y] \setminus \{0\}$, $f(0,0)=0$

Output: all possible Newton series $y = x^{q_1/p_1}\left(a_1 + x^{q_2/p_2}\left(a_2 + \dots\right)\right)$ that solve $f(x,y)=0$.

Recall Newton's algorithm for finding zeros of $\mathbb{R} \to \mathbb{R}$ functions.



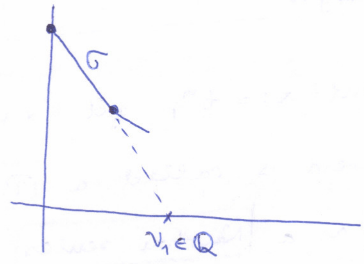
This is what we will imitate.

STEP 1. Let σ be the steepest segment of the Newton polygon of f

STEP 2. $q_1/p_1 := -1/\text{slope}(\sigma)$

STEP 3. Compute $f(x^{p_1}, x^{q_1}(a_1 + y_1))$.

$$f(x,y) = \underbrace{\sum_{p_1 i + q_1 j = p_1 \nu_1} a_{ij} x^i y^j}_{\substack{\text{quasi-homogeneous,} \\ \text{corresponds to } \sigma}} + \underbrace{\sum_{p_1 i + q_1 j > p_1 \nu_1} a_{ij} x^i y^j}_{\text{higher order terms}}$$



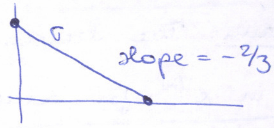
In $f(x^{p_1}, x^{q_1}(a_1+y_1))$, collect lowest order terms in x alone, and solve $f=0$ for a_1; there may be several solutions.

First approximation: $y = a_1 x^{p_1/q_1} + \dots$

STEP 4. $f_1(x, y_1) := f(x^{p_1}, x^{q_1}(a_1 + y_1)) \cdot x^{-p_1 \nu_1} \in \mathbb{C}[x, y_1]$

by construction, this is the highest power of x that can be factored out

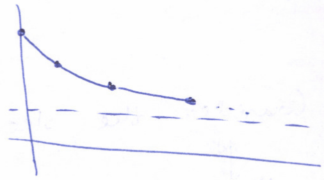
STEP 5. Repeat for f_1 instead of f. Obtain q_2/p_2, a_2, f_2, ...



Instructive example: $f(x,y) = y^2 - x^3$

$$\frac{q_1}{p_1} = \frac{3}{2} \qquad f(x^2, x^3(a_1+y_1)) = x^6(a_1+y_1)^2 - x^6 = x^6(a_2 - 1) + \text{higher order terms}$$

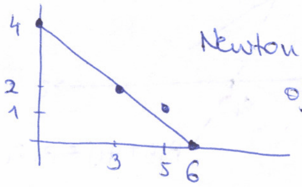
The only case when this doesn't work is when $y_n \mid f(x, y_n)$.
This means that the Newton polygon is above some horisontal line.



The different choices for a_i give the branches.

Note that the algorithm may be perfomed with any slope σ instead of the steepest one; Filip chose to present this version only out of cosmetic reasons.

More sophisticated example: $f(x,y) = (y^2-x^3)^2 - 4x^5y$



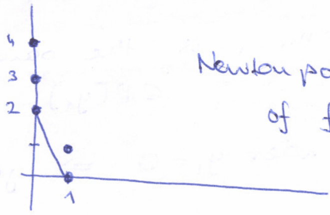
Newton polygon of f

$q_1/p_1 = 3/2 \qquad y = a_1 x^{3/2} + \ldots$

$$f(x^2, x^3(a_1+y_1)) = \left(x^6(a_1+y_1)^2 - x^6\right)^2 - 4x^{13}(a_1+y_1)$$
$$= x^{12}\cdot \underbrace{(a_1^2-1)^2}_{=0} + \text{higher order terms}$$

$\longrightarrow a_1 = \pm 1$, these give us two branches

$$V_1 = 6, \quad f_1(x, y_1) = x^{-12} f(x^2, x^3(\pm 1 + y_1))$$
$$= \left(\pm 2 \cdot y_1 + y_1^2\right) - 4x(\pm 1 + y_1)$$



Newton polygon of f_1

$q_2/p_2 = 1/2$

$$f_1(x^2, x^1(a_2+y_1)) = \left(\pm 2x(a_2+y_2) + x^2(a_2+y_2)^2\right)^2 \mp 4x^2 - 4x^3(a_2+y_2)$$
$$= 4x^2a_2^2 \mp 4x^2 + \text{higher order terms}$$

$$\Rightarrow a_2 = \begin{cases} \pm 1 & \text{if } a_1 = 1 \\ \pm i & \text{if } a_1 = -1 \end{cases} \qquad y = x^{3/2}\left(\pm 1 + x^{1/2 \cdot 2}\left(\begin{smallmatrix}\pm 1 \\ \pm i\end{smallmatrix} + \ldots\right)\right)$$

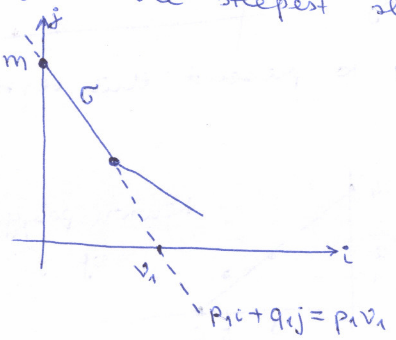
Claim. The denominators are bounded, i.e. Newton's algorithm gives a series in $x^{1/N}$ for some $N \in \mathbb{N}$.

Pf: $y = x^{q_1/p_1}\left(a_1 + x^{q_2/p_2}\left(a_2 + \dots\right)\right)$

We will show that $\exists i_0 : p_i = 1 \ \forall i \geq i_0$. Then take $N := p_1 \cdots p_{i_0}$.

Recall that $f_1(x, y_1) = x^{-p_1\nu_1} \cdot \underbrace{f\left(x^{p_1}, x^{q_1}(a_1 + y_1)\right)}_{\sum_{ip_1 + jq_1 = p_1\nu_1} a_{ij} x^{ip_1} y^{jq_1} (a_1+y_1)^j \; + \; \sum_{\dots > p_1\nu_1} \dots}$

Consider the steepest slope σ of the Newton polygon.



$m = \frac{p_1}{q_1}\nu_1 \qquad f(0, y) = y^m \cdot \text{const} + \text{h.o.t.}$

$$f_1(0, y) = \sum_{p_1 i + q_1 j = p_1\nu_1} a_{ij}(a_1 + y_1)^j + 0$$

$$= \text{const} \cdot y_1^{\frac{p_1\nu_1}{q_1}} + \text{lower order terms in } y_1$$

So if m_1 denotes the largest exponent of y_1 in $f_1(0, y)$ then $m_1 \leq \frac{p_1\nu_1}{q_1} = m$

By induction we obtain $m \geq m_1 \geq m_2 \geq \dots$

When do we have equality? It suffices to study $m_1 = m$.

$\underbrace{f_1(0, y_1)}_{\in \mathbb{C}[y_1]} = g(a_1 + y_1)$ for some polynomial $g(t) = \sum_{ip_1 + jq_1 = p_1\nu_1} a_{ij} t^j \in \mathbb{C}[t]$

Consider $y_1 = 0 \implies g(a_1) = 0 \quad (a_1 \neq 0$ by def$)$

$$\left.\begin{array}{l} \deg g = m = \frac{p_1\nu_1}{q_1} \\ m_1 = \left(\text{order of the zero } y_1 = 0 \text{ of } g(a_1 + y_1) \in \mathbb{C}[y_1]\right) \\ \quad = \left(\text{order of the zero } a_1 \text{ of } g(t) \in \mathbb{C}[t]\right) \end{array}\right\} \text{ equal by assumption}$$

$\implies (\text{const} \neq 0) \cdot (t - a_1)^m = g(t) = \sum_{p_1 i + q_1 j = p_1\nu_1} a_{ij} t^j$

coef of t^{m-1} is nonzero $\implies a_{i, m-1} \neq 0$ for some i s.t. $p_1 i + q_1(m-1) = p_1\nu_1$

$\implies p_1 i + p_1\nu_1 - q_1 = p_1\nu_1 \implies i = \frac{q_1}{p_1}$

But $i \in \mathbb{N}$ and $(p_1, q_1) = 1 \implies p_1 = 1$

Thus $p_1 \cdots p_{i+1}$ either does not increase (when $p_{i+1} = 1$)
or $m_{i+1} < m_i$

Knots, links & surfaces

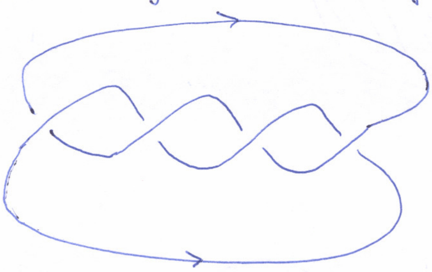
14.05.2019

This will be a non-comprehensive, non-precise introduction, just to give us some vocabulary to work with.

Def. $\boxed{\text{Knot}}$: oriented, smoothly embedded $S^1 \hookrightarrow S^3$.

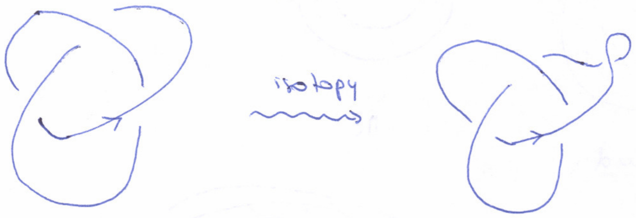
$$S^3 = \mathbb{R}^3 \cup \{\infty\} = \{(x,y) \in \mathbb{C}^2 \mid |x|^2 + |y|^2 = 1\} = \{p \in \mathbb{R}^4 \mid \|p\| = 1\}$$

Def. $\boxed{\text{Link}}$: disjoint union of fin many knots: $S^1 \amalg \ldots \amalg S^1 \hookrightarrow S^3$



Def. $\boxed{\text{Isotopy}}$: $\varphi : [0,1] \times S^3 \longrightarrow S^3$ continuous, $\varphi_t(p) := \varphi(t,p)$, $\varphi_t : S^3 \longrightarrow S^3$ is a homeomorphism $\forall t \in [0,1]$, $\varphi_0 = id_{S^3}$

Def. $K_0, K_1 \hookrightarrow S^3$ knots are $\boxed{\text{isotopic}}$ if $\exists \varphi$ isotopy s.t. $\varphi_1(K_0) = K_1$.

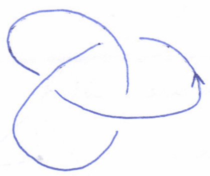


$$\begin{tikzcd} K \arrow[r, hook] \arrow[dr, dashed] & S^3 \setminus \{\text{point}\} \cong \mathbb{R}^3 \arrow[d, "\text{projection}"] \\ & \mathbb{R}^2 \end{tikzcd}$$

PO

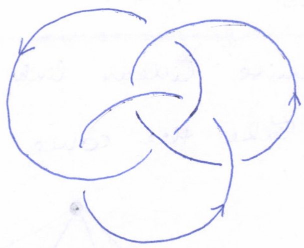
We can do this so that there are no triple crossings, every crossing looks like

Ex.

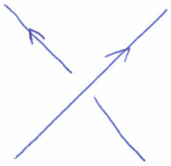


Trefoil knot

Figure 8 knot

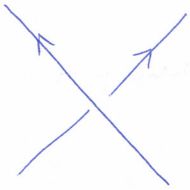


Borromean rings



$+$

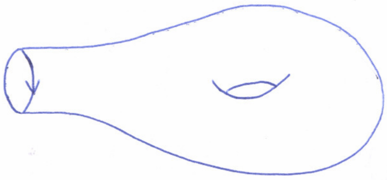
Positive crossing



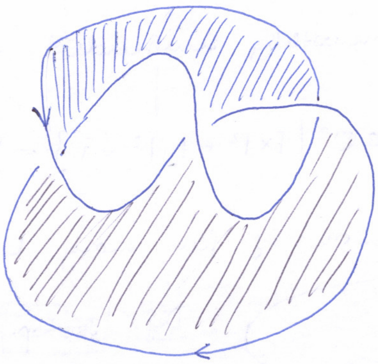
$-$

Negative crossing

Def. $\boxed{\text{Seifert surface}}$: compact, connected, oriented surface with boundary $S \hookrightarrow S^3$.

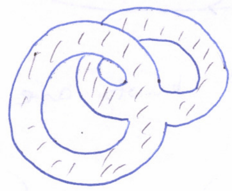


Seifert surface that has the unknot as its boundary



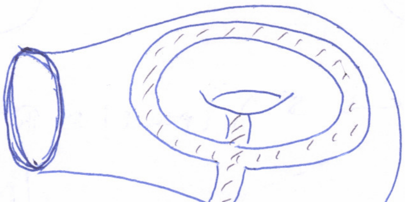
S

These surfaces are homeomorphic:

$S \cong$  $\cong$  $\cong$ 

Hopf band

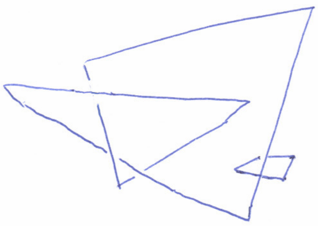
$\cong$



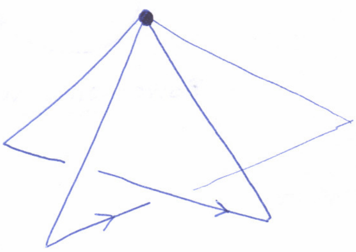
But they are not isotopic since their boundaries are not isotopic (let's just believe that for now.)

Thm. (Frankl–Pontrjagin, Seifert) $\forall L \hookrightarrow S^3$ link $\exists S \hookrightarrow S^3$ Seifert surface: $\partial S = L$.
1929, 1934

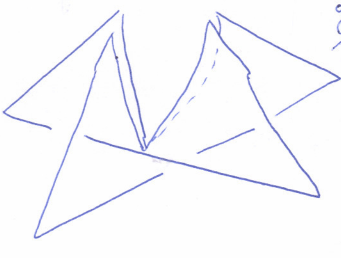
Pf: Consider a piecewise linear link "near" a plane



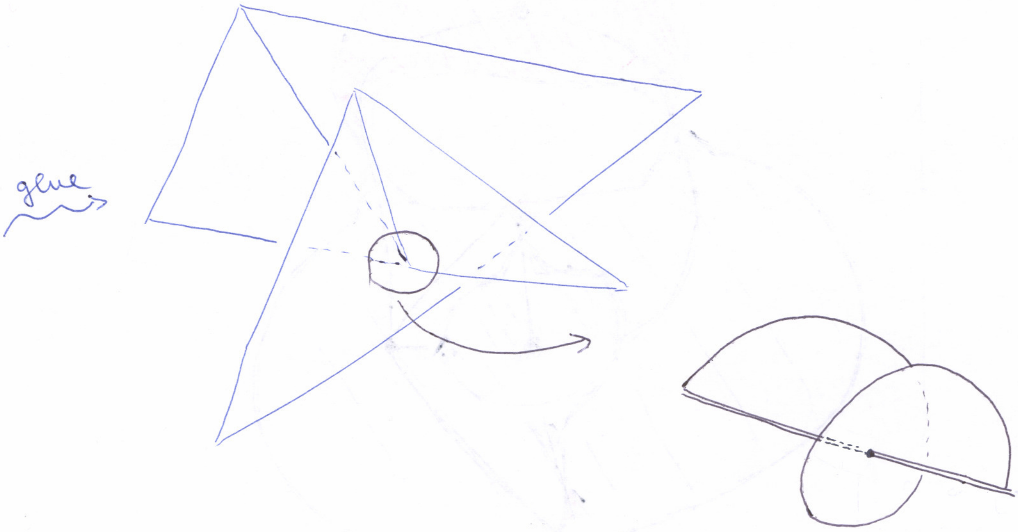
Take the cone $\longrightarrow$ obtain a cell complex.



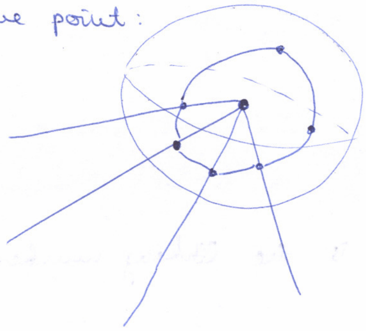
cut off the tip



glue

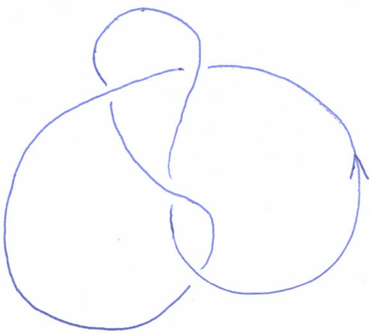


Near the cone point:

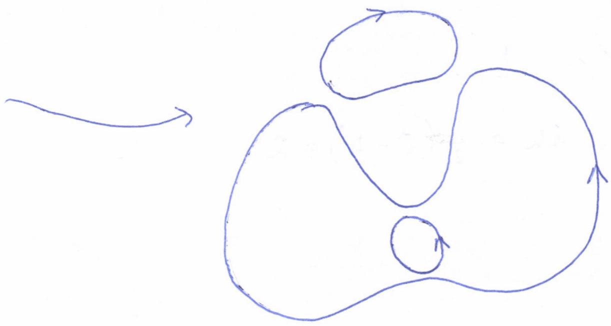


remove the ball and glue in disjoint discs.

<u>Seifert's PROOF for the Thm.</u> through an example:

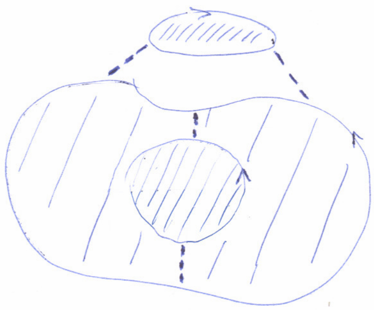


Always replace $\times$ and $\times$ by)(

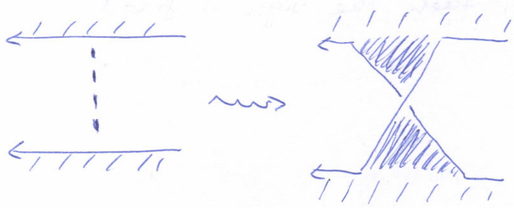


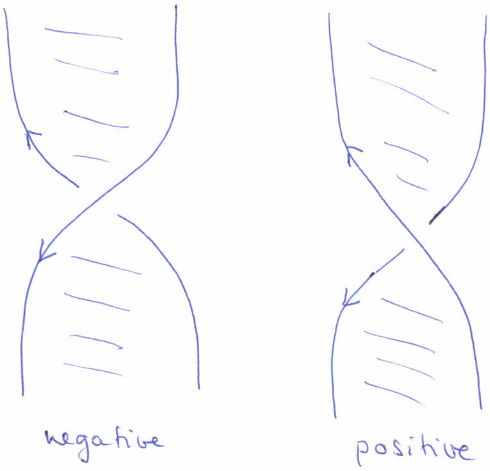
Thus we obtain a disjoint union of oriented circles.

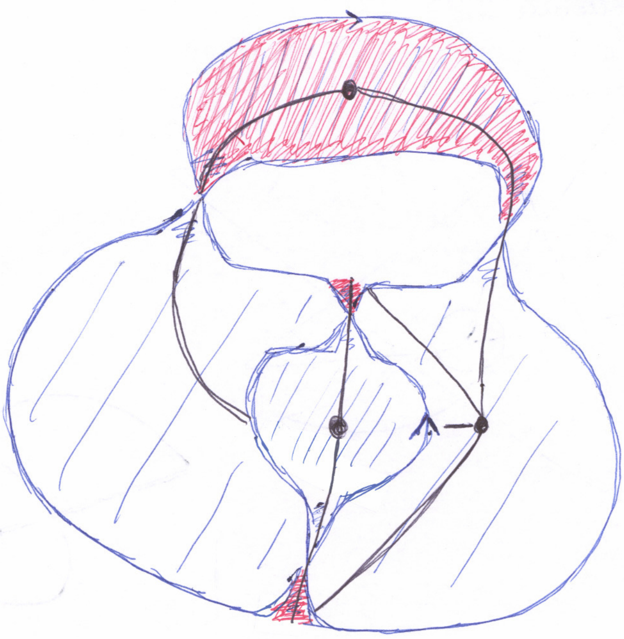
We place each circle on a separate plane, these planes being above one another. Use Jordan's Thm on each of these planes.
$\rightarrow$ discs



For each former crossing, glue in a half-twisted band:

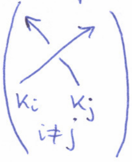
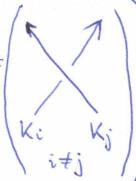




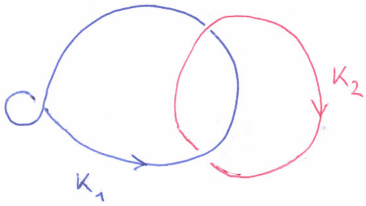


Linking number

Let $K_1, K_2 \hookrightarrow S^3$ be disjoint knots

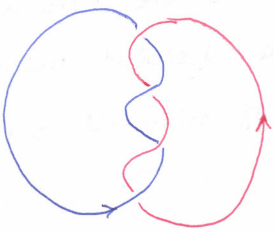
Def. $\boxed{lk(K_1, K_2)} := \frac{1}{2}\Bigg(\#\Big($  $\Big) - \#\Big($  $\Big)\Bigg)$ is the linking number of K_1, K_2.

Ex.



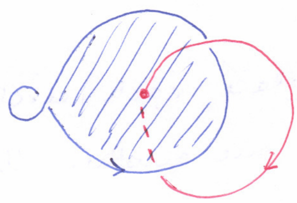
$lk(K_1, K_2) = \frac{1}{2}(2-0) = 1.$

Ex.



$lk = \frac{1}{2}(0-4) = 2$

Rmk. $lk(K_1, K_2) = \#(S \cap K_2)$ where $K_1 = \partial S$.



Rmk. $lk(K_1, K_2) = \pm [K_2] \in H_1(S^1 \setminus K_1; \mathbb{Z}) \simeq \mathbb{Z}$

$= \pm [K_1] \in H_1(S^2 \setminus K_2; \mathbb{Z}) \simeq \mathbb{Z}$

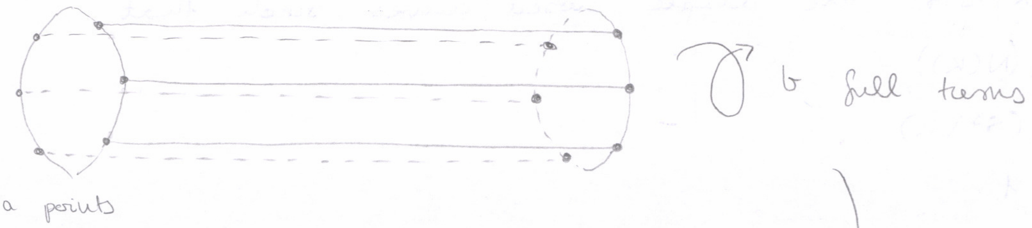
choose a generator once and for all, then the sign is fixed

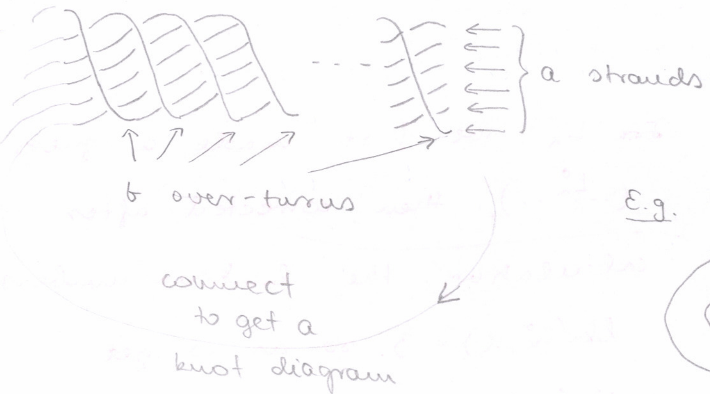
Recall that a torus is a surface homeomorphic to $S^1 \times S^1$.

<u>Def.</u> A $\boxed{\text{torus knot}}$ K is a knot which can be drawn (i.e. embedded) on the surface of a standardly embedded torus. (into S^3)

<u>Def.</u> For $(a,b) \in \mathbb{Z}^2$, $(a,b)=1$ let $\boxed{T(a,b)}$ be the torus knot representing the class $(a,b) \in \mathbb{Z} \oplus \mathbb{Z} \simeq H_1(S^1 \times S^1; \mathbb{Z})$.

In other words, $T(a,b)$ wraps around a times in one direction, b times in the other.





<u>E.g.</u> $(a,b) = (2,3)$

Recall that homology also sees orientation.

<u>Rmk.</u> $\# \pi_0\left(\{y^n - x^m = 0\} \cap S^3\right) = \gcd(n,m)$. This is why we assume $(a,b)=1$.

<u>Rmk.</u> $T(a,b) = T(b,a)$, and $T(\pm 1, b) =$ unknot

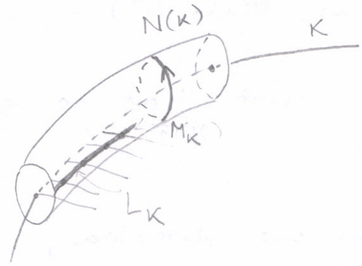
Take tubular nbhds:

$S^1 \hookrightarrow S^3$ gives rise to $S^1 \times D^2 \hookrightarrow S^3$

Meridian and longitude

Let $K \hookrightarrow S^3$ be a knot. Take a tubular nbhd of K in S^3:

formally $K: S^1 \hookrightarrow S^3$ induces $\boxed{N(K)}: S^1 \times \mathbb{D}^2 \hookrightarrow S^3$ s.t. $N(K)(S^1 \times \{0\}) = K(S^1)$



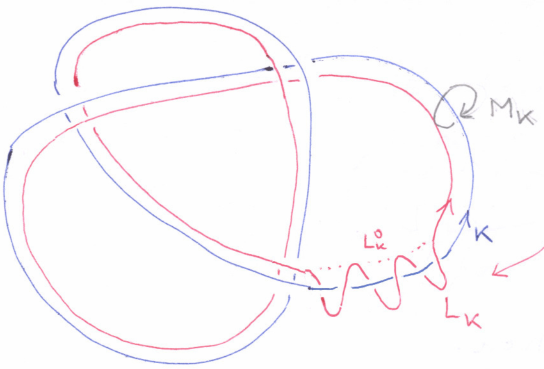
Rmk. $\partial N(K) \cong S^1 \times S^1$.

Rmk. We identify the embeddings K, $N(K)$ with their images.

Def. $\boxed{M_K}, \boxed{L_K} \subset \partial N(K) \subset S^3$ are simple closed curves such that

- $[M_K] = 1 \in H_1(S^3 \setminus K)$
- $[L_K] = 0 \in H_1(S^3 \setminus K)$
- $lk(M_K, L_K) = 1$
- $[L_K] = [K] \in H_1(N(K))$

Ex. Trefoil knot



For L_K, we first made a guess $(\ldots L_K^0 \ldots)$, then corrected after calculating the linking numbers $lk(L_K^0, K) = 3$, so as to get $lk(L_K, K) = 0$.

Cable of a knot

$K \hookrightarrow S^3$, $\varphi: S^1 \times S^1 \xrightarrow{\cong} \partial N(K)$

$S^1 \times \{1\} \longrightarrow L_K$

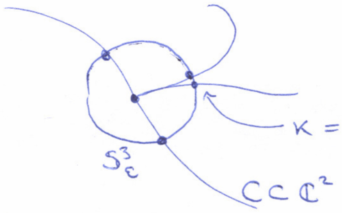
$\{1\} \times S^1 \longrightarrow M_K$

Def. For $(a,b) \in \mathbb{Z}^2$, the $\boxed{(a,b)\text{-cable}}$ of K is the knot / link

$\boxed{K_{(a,b)}} := \varphi(T(a,b)) \subset \partial N(K) \subset S^3$

Explicitly: $K_{(a,b)} = a L_K + b M_K \in H_1(\partial N(K))$.

Recall: we wish to describe the intersection of S^3_ε with a curve C around a singularity.

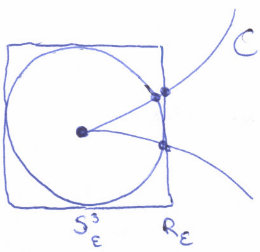


Theorem. Let $y = x^{q_1/p_1}\left(a_1 + x^{q_2/p_1p_2}\left(a_2 + \dots + a_s x^{q_s/p_1\cdots p_s}\right)\dots\right)$ be the Newton series parametrising a branch of an algebraic curve C near $(0,0)$.
Then for sufficiently small $\varepsilon > 0$ the knot $K = S^3_\varepsilon \cap C$ is
the (p_s, α_s)-cable of the (p_{s-1}, α_{s-1})-cable of $\dots$ of the (p_1, α_1)-cable of the unknot $\bigcirc$
where $\alpha_1 := q_1$, $\alpha_{i+1} := q_{i+1} + p_{i+1} p_i \alpha_i \quad \forall i \geq 1$.

If $(p_i, \alpha_i) = 1$ s.t. $p_i > 0$, $\alpha_1 > 0$, $\alpha_{i+1} > p_{i+1} p_i \alpha_i$ then we can reconstruct (p_i, q_i) s.t. $p_i, q_i > 0$, $(p_i, q_i) = 1$.

$\Rightarrow$ The knot of the singularity is $\left(\left(\left(\bigcirc_{(p_1,\alpha_1)}\right)_{(p_2,\alpha_2)}\right)_{(p_3,\alpha_3)} \dots\right)_{(p_s,\alpha_s)}$

PF OF THM: Replace S^3_ε by $R_\varepsilon := \{(x,y) \in \mathbb{C}^2 \mid |x| = \varepsilon,\ |y| \leq \varepsilon\} \cong S^1 \times \mathbb{D}^2$



(We will explain why this is possible later.)
Consider the i^{th} approximation of y:

$$y_i = x^{q_1/p_1}\left(a_1 + x^{q_2/p_1p_2}\left(a_2 + \dots + x^{q_i/p_1\cdots p_i}\right)\dots\right)$$

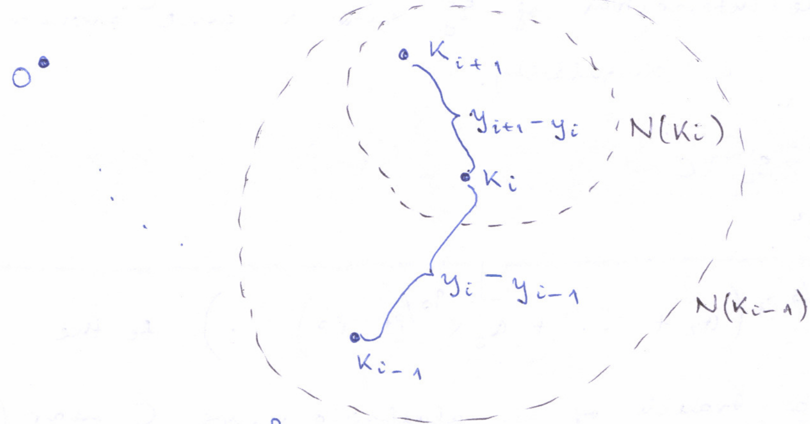
This defines a knot $K_i \subset R_\varepsilon$.

Induction on i:

- $y_1 = a_1 x^{q_1/p_1}$ describes the torus knot $K_1 = T(p_1, q_1)$; by an exercise, this is the (p_1, q_1)-cable of $\bigcirc$.
- i to $i+1$: study K_j for $j = i-1, i, i+1$.

x fixed, $|x| = \varepsilon$.

The plane of the paper sheet is the y-disk:



$$y_i - y_{i-1} = a_i\, x^{\beta_i}$$

$$\beta_i = \frac{q_i}{p_1\cdots p_i} + \frac{q_{i-1}}{p_1\cdots p_{i-1}} + \dots + \frac{q_1}{p_1}$$

$$|y_{i+1}-y_i| = a_{i+1}\, x^{\beta_i}\, x^{q_{i+1}/p_1\cdots p_{i+1}} = \underbrace{\left|\frac{a_{i+1}}{a_i}\right|}_{\ll |y_i - y_{i-1}|} \left|(y_i - y_{i-1})\right| \cdot \underbrace{\left|x^{q_{i+1}/p_1\cdots p_{i+1}}\right|}_{\text{power of } \varepsilon}$$

$\Rightarrow$ we can choose tubular nbhds $N(K_{i-1})$, $N(K_i)$ such that

$$K_i \subset N(K_i) \subset N(K_{i-1}) \setminus K_{i-1}$$

Parametrise the knot: $x = \varepsilon\, t^{p_1\cdots p_i}$ for $t \in S^1$

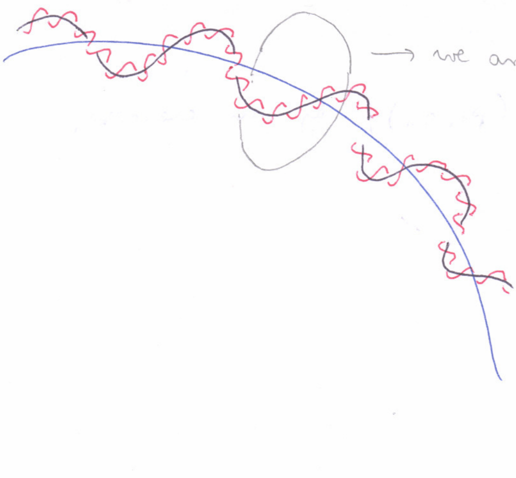
For x fixed, we get $p_1\cdots p_i$ values for t.

$$y_{i+1} - y_i = \varepsilon\,(y_i - y_{i-1})\cdot t^{q_{i+1}/p_{i+1}}$$

When t makes a full turn, $y_1, \dots, y_i$ all come back to the same position.

$(y_i - y_{i-1})$ makes some number of full turns;

$(y_{i+1} - y_i)$ makes the same turns as $(y_i - y_{i-1})$, plus $\frac{q_{i+1}}{p_{i+1}}$ of a turn.



$\to$ we are currently looking at a cross-section of a circular motion around a circular motion around a circular motion ...

(Think of the Sun, the Earth and the Moon.)

For x fixed, we have $p_1\cdots p_i$ values for y_i.

For each of these y_i-values, there are p_{i+1} values for y_{i+1}.

Computing the cabling coeffs:

Let M_j, L_j be the meridian and (preferred, i.e. determined by Seifert surface) longitude of K_j.

Induction hyp.: $K_i = p_i L_{i-1} + \alpha_i M_{i-1} \in H_1(N(K_{i-1}) \setminus K_{i-1})$

Let L be the knot obtained by pushing K_i away from K_{i-1} by some small distance $\delta > 0$.

$y_L = y_i + \delta x^{\beta_i}$ $\qquad y_i = y_{i-1} + a_i x^{\beta_i}$

$K_{i+1} = p_{i+1} L + q_{i+1} M_i$

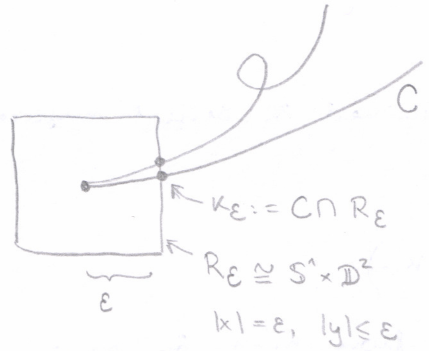
Wts: $L = L_i + p_i \alpha_i M_i$

From this: $K_{i+1} = p_{i+1} L_i + \overbrace{(q_{i+1} + p_{i+1} p_i \alpha_i)}^{\alpha_{i+1}} M_i$, which we wanted to prove.

$$\left.\begin{array}{l} \alpha_i = lk(K_i, K_{i-1}) = lk(L, K_{i-1}) \\ L = K_i \in H_1(N(K_i)) \end{array}\right\} \Rightarrow L = 1 L_i + \alpha_i M_{i-1} \in H_1(N(K_{i-1}) \setminus (K_{i-1} \cup K_i))$$

$$M_{i-1} = p_i M_i$$

Recall what we did last week:



- K_ε is a link which is independent of ε for ε sufficiently small.
- K_ε is an iterated cable of the unknot, with cabling coeffs only depending on the Newton pairs of the branches.

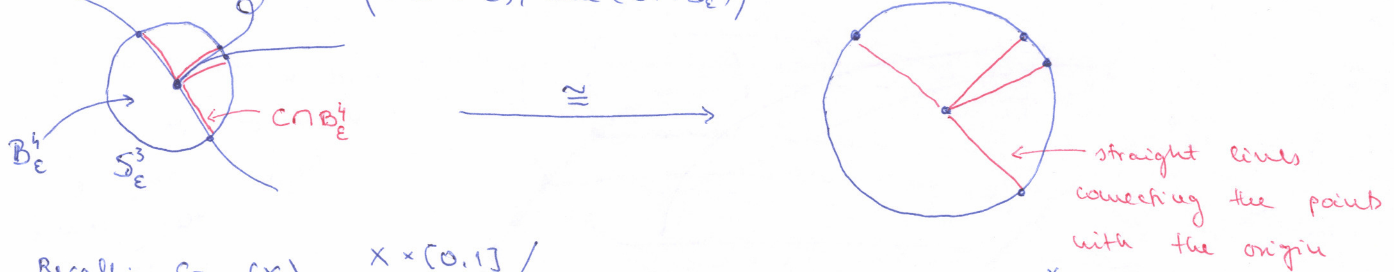
(Tbh we only looked at 1 branch, we didn't investigate what happens when multiple branches are present.)

Summary of all this:

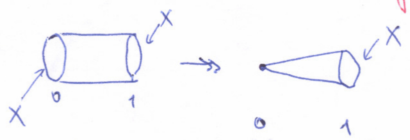
Thm. (Conical structure) $C \subset \mathbb{C}^2$ algebraic curve, $(0,0) \in C$. Then $\exists \varepsilon_0 > 0$ s.t. $\forall \varepsilon \leq \varepsilon_0$ s.t. 1) $K_\varepsilon := C \cap S^3_\varepsilon \subseteq S^3_\varepsilon \cong S^3$ is a link indep of ε

2) $\left(\mathbb{B}^4_\varepsilon, C \cap \mathbb{B}^4_\varepsilon\right) \cong \mathrm{Cone}\left(S^3_\varepsilon, C \cap S^3_\varepsilon\right)$ homeomorphic as pairs of top spaces

$:= \left(\mathrm{Cone}\,(S^3_\varepsilon), \mathrm{Cone}\,(C \cap S^3_\varepsilon)\right)$



Recall: $\mathrm{Cone}(X) := X \times [0,1] \Big/ \{(x,0) \sim (y,0) \mid \forall x, y \in X\}$

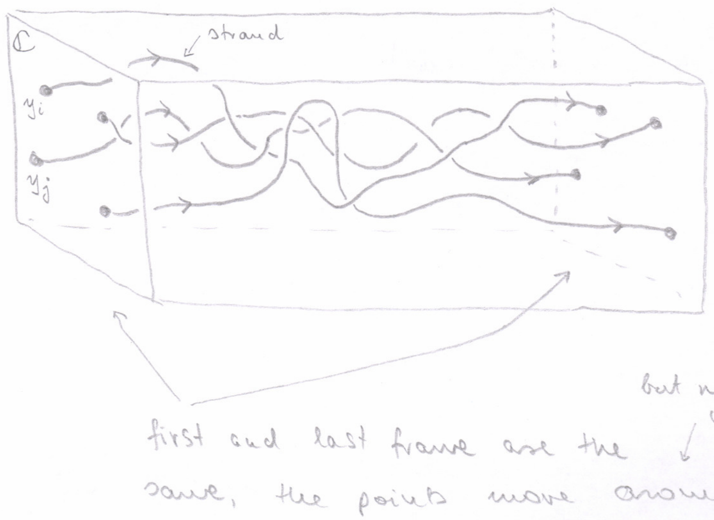


In words: the link of a singularity determines the local embedded topology of the curve C near the singular point.

Thm. Links of singularities can be represented by positive braid diagrams.

$P_n := \{ p \in \mathbb{C}[y] \mid \deg p = n \text{ and } p \text{ has } n \text{ distinct roots} \}$

$B_n := \pi_1(P_n)$ braid group on n strands



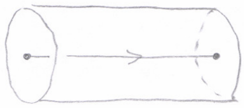
first and last frame are the same, the points move around, but they need not arrive to the same spot they started from

but never coincide!

Algebraic Curves, lecture 9

SKETCH OF PROOF:

We do induction on the number of cabling operations.

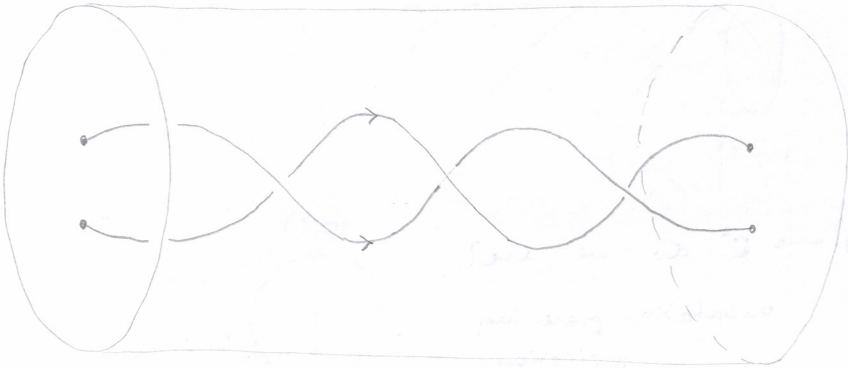


1^{st} iteration $\longrightarrow$ ✓

$$y_i - y_{i-1} = a_i x^{\beta_i}$$

Induction step:

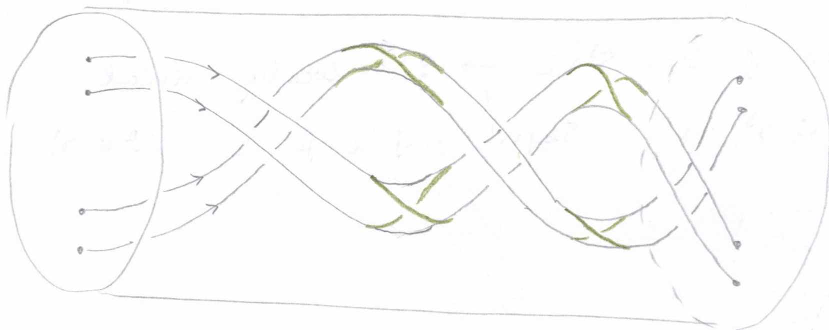
K_i



K_i : i^{th} iteration of a positive braid

all crossing are $+$

K_{i+1}

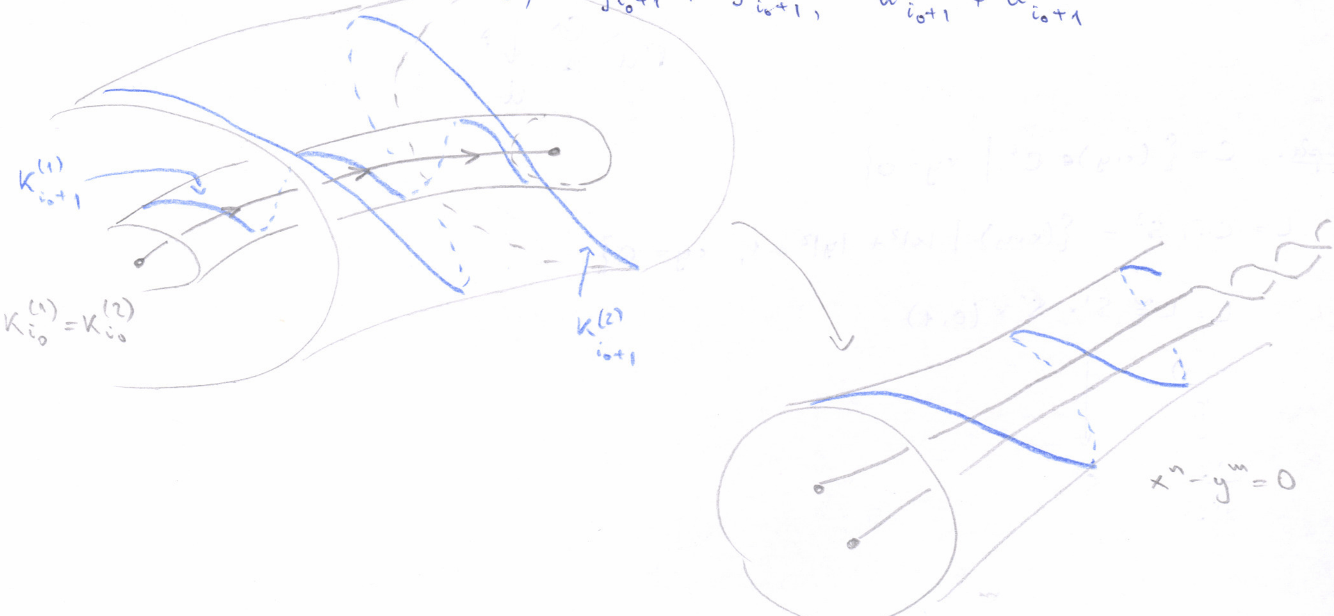


all crossings are still $+$

For each pos crossing of K_i, we get p_{i+1}^2 pos crossings for K_{i+1}, plus additional terms.

$y_i^{(1)}, y_i^{(2)}$: two Newton series for the two distinct branches

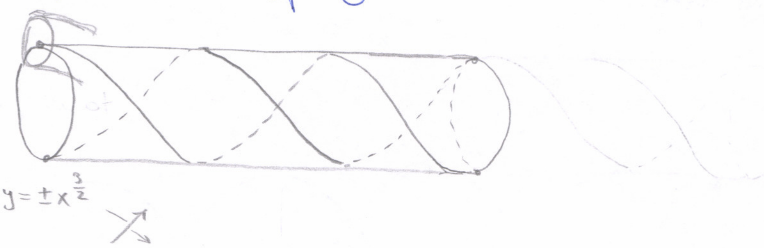
$y_i^{(1)} = y_i^{(2)}$ for $i = 1, \dots, i_0$; $y^{(1)}_{i_0+1} \neq y^{(2)}_{i_0+1}$, $a^{(1)}_{i_0+1} \neq a^{(2)}_{i_0+1}$



Example. $y^{(1)} = x^{\frac{3}{2}}\left(1 + x^{\frac{1}{4}}\right) = x^{\frac{3}{2}} + x^{\frac{7}{4}}$ $\Big|$ $y^{(2)} = x^{\frac{3}{2}}\left(1 + x^{\frac{1}{6}}\right) = x^{\frac{3}{2}} + x^{\frac{10}{6}}$

$$K^{(1)} := (p_2, \underbrace{\alpha_2}_{p_2 p_1 \alpha_1 + q_2})\text{-cable of } (\underbrace{p_1}_{2}, \underbrace{\alpha_1}_{q_1 = 3})\text{-cable of } \bigcirc$$

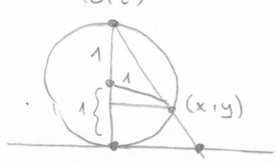
$$= (3, 19)\text{-cable of } (2,3)\text{-cable of } \bigcirc$$

$\bigcirc_{(2,3)} =$  

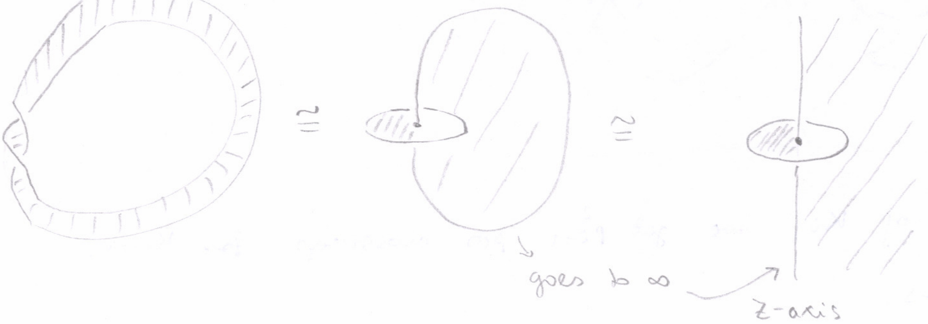
Issue. Which embedding $S^3 \hookrightarrow \mathbb{C}^2$ do we use?

$rS^1 \times sS^1 \subset S^3 \setminus \{(0,i)\} \longrightarrow \mathbb{R}^3$ orientation-preserving projection

$$(a+ib, c+id) \longmapsto \frac{2}{1-d}(a,b,c)$$



Def. A link $L \subset S^3$ is fibred if $\exists p : S^3 \setminus L \longrightarrow S^1$ locally trivial fibre bundle s.t. $p^{-1}(\vartheta) \cup L \subset S^3$ is a Seifert surface for L $\forall \vartheta \in S^1$.



Recall. $p : E \longrightarrow B$ is a locally trivial fibre bundle with fibre F if $\forall b \in B$ $\exists U \subseteq B$ open nbh of b s.t.

$$\begin{tikzcd} U \times F \arrow[r, "\cong"] \arrow[dr, "pr_U"'] & p^{-1}(U) \subset E \arrow[d, "p"] \\ & U \end{tikzcd}$$

Example. $C = \{(x,y) \in \mathbb{C}^2 \mid xy = 0\}$

$$L = C \cap S^3 = \{(x,y) \mid |x|^2 + |y|^2 = 1,\ xy = 0\}$$

$$S^3 \setminus L \cong S^1 \times S^1 \times (0,1)$$

$$\begin{array}{cc} (x,y) & \\ \downarrow & \downarrow \\ \frac{xy}{|xy|} & S^1 \end{array}$$

<u>Thm.</u> (Milnor) Links of singularities are fibred:

$$L = C \cap S^3_\varepsilon, \quad C = \{(x,y) \in \mathbb{C}^2 \mid f(x,y) = 0\}$$

$$\frac{f}{|f|} : S^3_\varepsilon \setminus L \longrightarrow S^1 \quad \text{is a locally trivial fibration.}$$

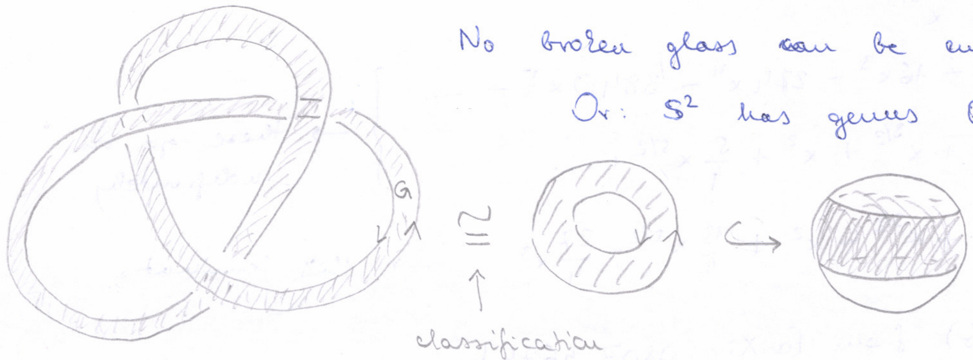
$$q \longmapsto \frac{f(q)}{|f(q)|} \qquad (f(q) \neq 0 \text{ since } q \notin L)$$

04.06.2019

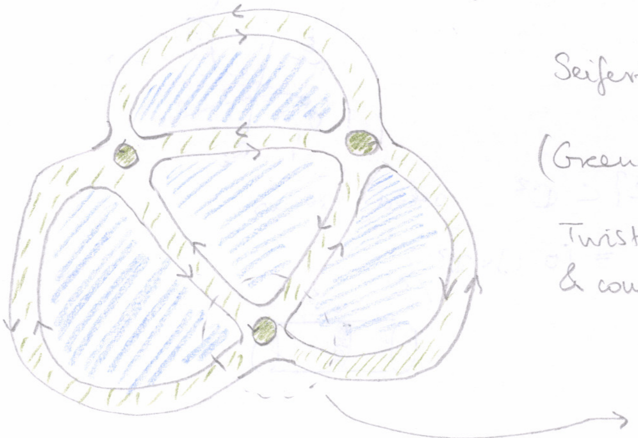
<u>Exercise 22.</u> The ribbon b/w the two knots is a suitable Seifert surface.

No broken glass can be embedded $\rightarrow g = 0$.

Or: S^2 has genus 0, and this surface can be embedded into S^1



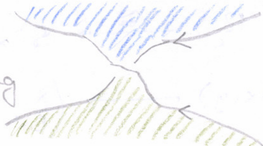
classification of surfaces



Seifert: replace $\times$ by $)($

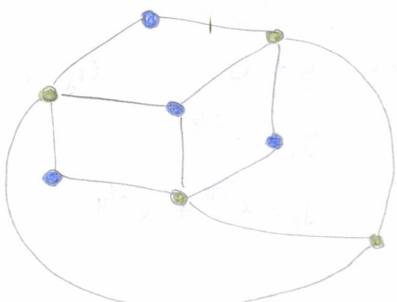
(Green: counterclockwise; blue: clockwise)

Twisting & connecting



Then either embed broken glasses to compute the genus,

or use Euler: retract to the graph



$$\left.\begin{array}{l} \chi \overset{\text{def}}{=} 8 - 12 = -4 \\ \chi = 2 - 2g - \underbrace{b}_{2} \end{array}\right\} \Rightarrow g = 2$$

Question. Does the Newton algorithm always terminate?

Recall: we stop only if $y_n \mid f_0$.

This may never happen. But this need not trouble us: for i large enough, $p_i = 1$, and taking the cable is just an isotopy, hence the knot type does not change.

Note: we have convergence due to WPT.

Answers to 2(a): $y = x^{3/2} + x^{5/3}$

b): $y = -2x^2 - 16x^3 - 224x^4 - 3840x^5 - \dots$

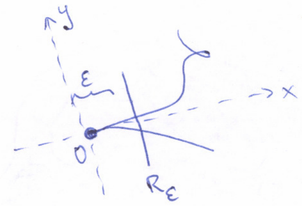
$y = -\frac{1}{2}x + x^{3/2} + x^2 + \frac{5}{4}x^{5/2} + \dots$

$\Big\} \longrightarrow$ these go on indefinitely

$y = -\frac{1}{2}x - x^{3/2} - x^2 + \frac{13}{4}x^{5/2} - \frac{27}{4}x^3 \longrightarrow$ this terminates

a) is from [Ghys], b) from [arXiv: 0807.4674]

Addendum to the Thm on p. 25



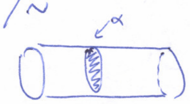
$C \subset \mathbb{C}^2$ branch of an alg curve

$C \cap R_\varepsilon \cong C \cap S^3_\varepsilon$

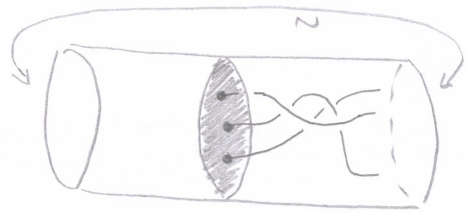
$R_\varepsilon = \{ |x| = \varepsilon,\ |y| \le \varepsilon \} \subset \mathbb{C}^2$ solid torus

$R_\varepsilon \xrightarrow{\cong} S^1 \times \mathbb{D}^2 = [0,1] \times \mathbb{D}^2 /_\sim$

$(x,y) \longmapsto \left(\frac{x}{\varepsilon}, \frac{y}{\varepsilon}\right)$



$x = \varepsilon \cdot e^{2\pi i \alpha}$, fixing x means fixing α



$y_1 = a_1 x^{q_1/p_1}$

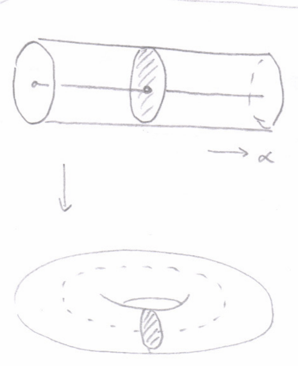
$y_2 = x^{q_1/p_1}\left(a_1 + x^{q_2/p_1 p_2}\, a_2\right)$

We give an example of how this goes.

$y = x^{3/2} + x^{7/4} \rightsquigarrow$ approximations: $y = 0$ (if x is small, y is even smaller)

$y_1 = x^{3/2}$

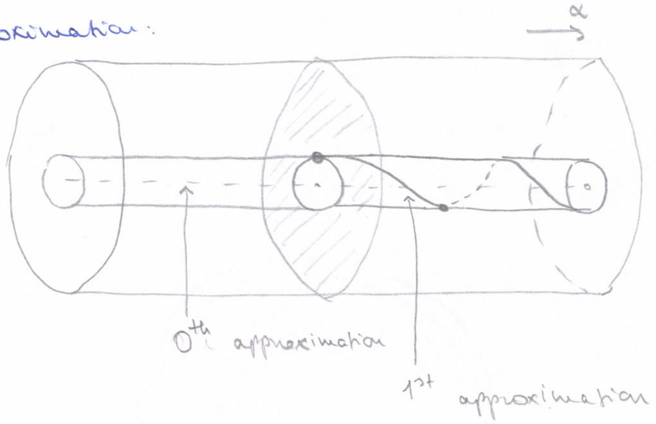
$y_2 = x^{3/2} + x^{7/4}$



$x = \varepsilon e^{2\pi i \alpha}$

0th approximation

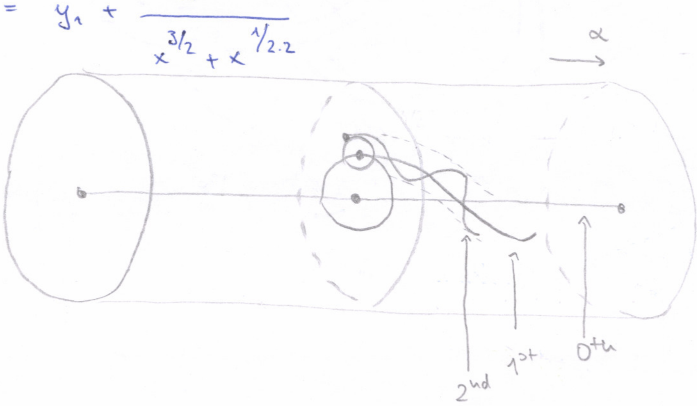
1^{st} approximation:



$\to$ we get a torus knot

The error is small compared to the previous step.

$$y_2 = y_1 + \frac{x^{7/4}}{x^{3/2} + x^{1/2 \cdot 2}}$$

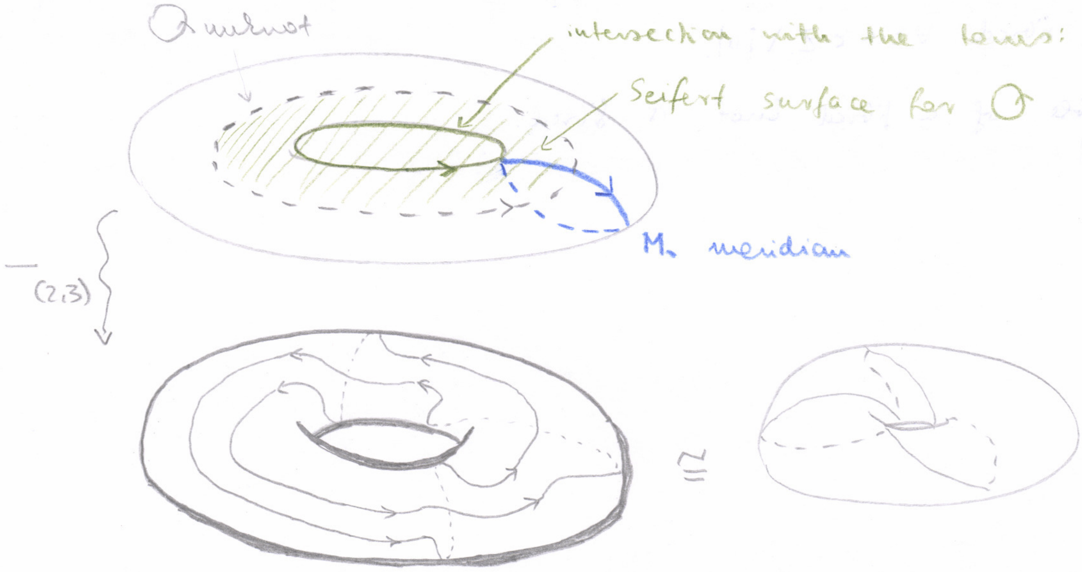


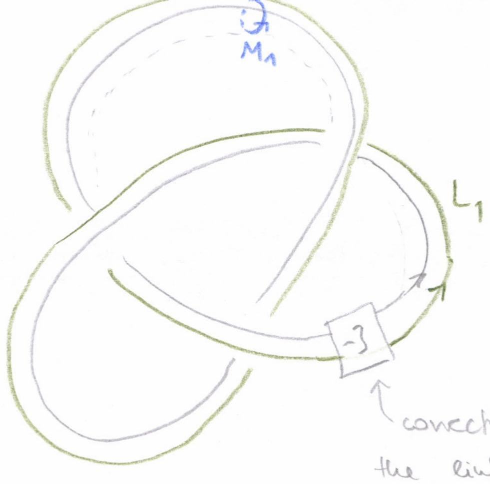
How many times we have to go around to get back to the same point is encoded in the exponents of x.

$y = x^{3/2}\left(1 + x^{1/2 \cdot 2}\right)$ $\qquad \frac{q_1}{p_1} = \frac{3}{2}$ $\qquad \frac{q_2}{p_2} = \frac{1}{2}$ $\qquad \alpha_1 = q_1 = 3$

$\alpha_2 = q_2 + p_2 p_1 \alpha_1 = 13$

$$K = C \cap R_\varepsilon = \left(O_{(2,3)}\right)_{(2,13)}$$





correcting s.t. the linking number is 0

Want to draw $2L_1 + 13M_1$.

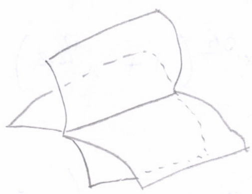
$$lk = \frac{12 + ?}{2} = 13$$

<u>Theorem.</u> Algebraic knots are fibred. 25.06.2019

Recall: $K = S^3 \cap C$ algebraic knot

$C \subset \mathbb{C}^2$ algebraic curve, S^3 a small sphere

K is fibred if $\exists p : S^3 \setminus K \longrightarrow S^1$ fibre bundle s.t. $F_t := p^{-1}(t) \cup K$ is a Seifert surface $\forall t \in S^1$

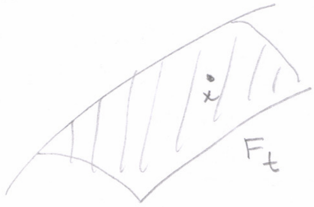


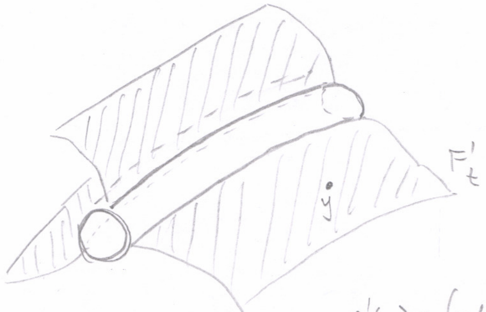
Pf: We know that alg knots are iterated cables of O.

We will show the general statement that

if K is fibred, $p: S^3 \setminus K \longrightarrow S^1$ the fibre bundle s.t. $\forall t \in (0,1]$: $p^{-1}(t) \cup K = F_t$ is Seifert then $K' := K_{(a,b)}$ is fibred $\forall a, b \in \mathbb{Z} \setminus \{0\}$.

That is, the cable of a fibred knot is fibred.





For $y \in S^3 \setminus N(K)$, define p' by:

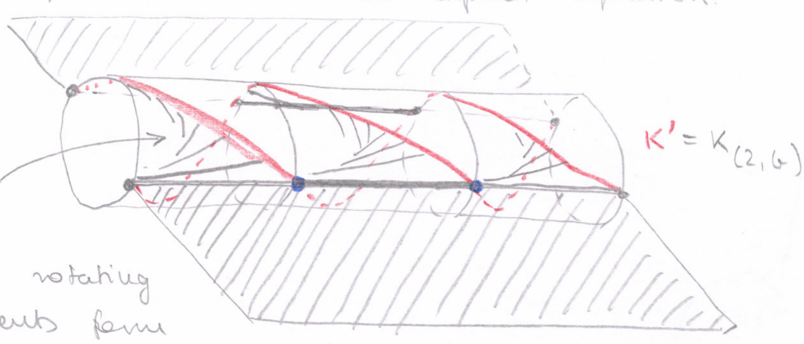
$p(x) = e^{2\pi i t}$

$$p'(y) = \left(p(y)\right)^a = e^{2\pi i t}$$

p' will be a map $S^3 \setminus K' \longrightarrow S^1$, so far we have defined it only outside $N(K)$.

For $y \in N(K)$, we won't have an explicit equation.

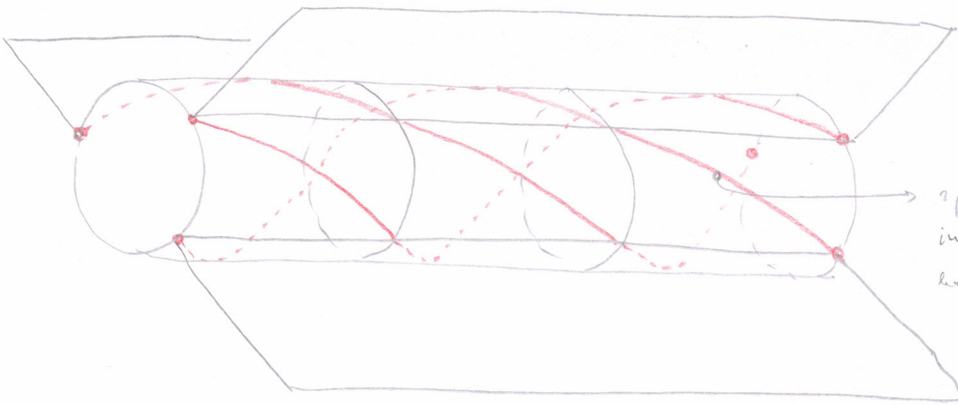
a=2



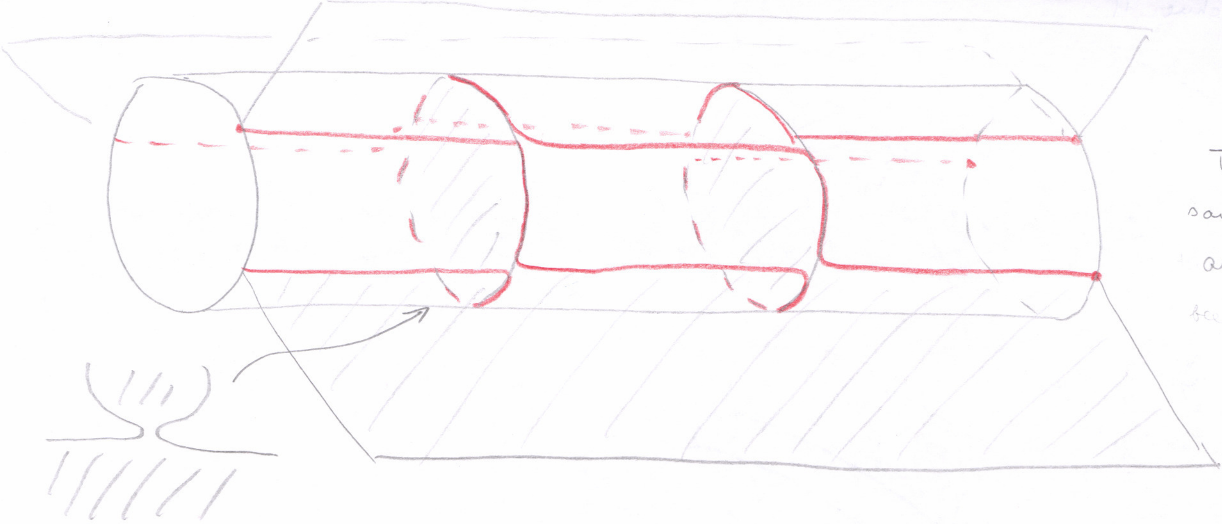
these rotating segments form the fibre bundle $\rightarrow$ disjoint surfaces except for the gluing pts

Blue: gluing points, around these: 

a=3

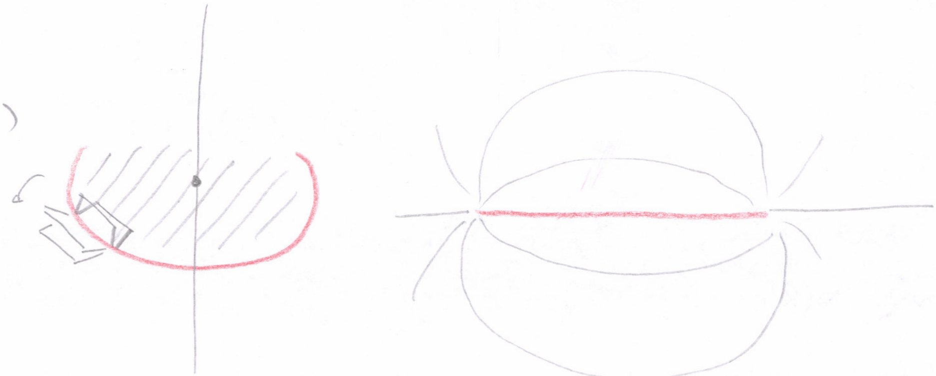


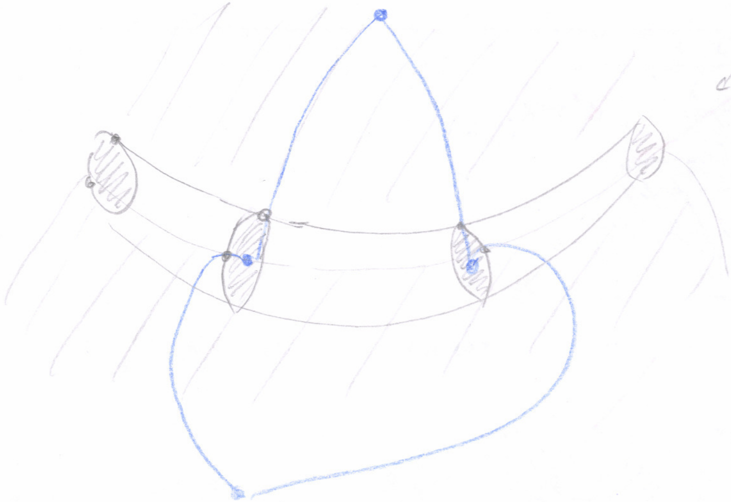
split these arcs in half, push one half downwards, the other one upwards to get the next picture



This is the same picture as the previous one

$T(2,3) = \bigcirc_{(2,3)}$

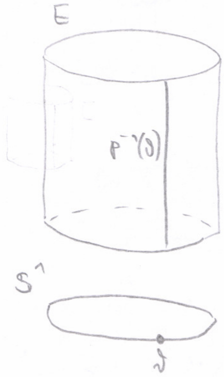




what the hell is this?

Monodromy

$$\begin{array}{ccc} F & \hookrightarrow & E \\ & & \downarrow p \\ & & S^1 \end{array} \quad \text{fibre bundle}$$

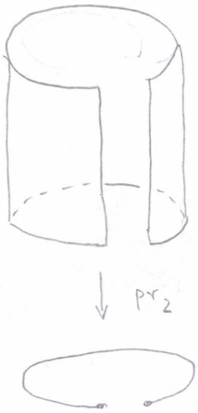


$p^{-1}(\vartheta) \cong F$

In our case: $E = S^3 \setminus K$

$F =$ Seifert surface of K

Cutting E open:



$F \times [0,1]$

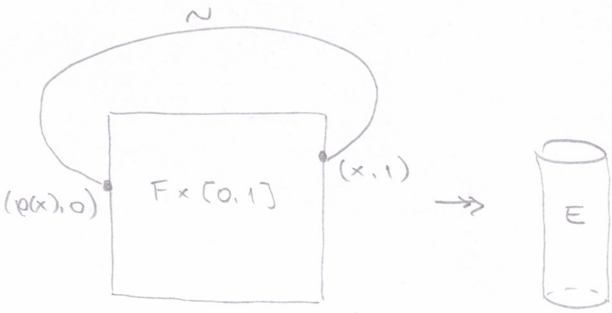
$[0,1]$

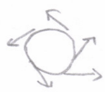
$\rightarrow$ we get a bundle over $[0,1]$

The fibre bundle $p : E \rightarrow S^1$ is thus described by a map $\varphi : F \longrightarrow F$ called the monodromy of p.

(Note that φ need not be unique.)

$$\begin{array}{c} E \cong F \times [0,1] \Big/ (x,1) \sim (\varphi(x), 0) \\ \downarrow \quad \swarrow G \qquad (x,t) \\ p \qquad \swarrow \\ e^{2\pi i t} \end{array}$$





Constructing φ: let $\vartheta_z := 2\pi i z$, this gives a vector field on $S^1 \subset \mathbb{C}$

$\Theta :=$ a lift of ϑ to $S^3 \setminus K$

$$(dp)_x : T_x S^3 \longrightarrow T_{p(x)} S^1$$

$$T_x S^3 \cong \underbrace{T_x F_{p(x)}}_{p \text{ does not vary here, } dp = 0} \oplus N_x F_{p(x)}$$

$$\Theta_x = \begin{cases} \left((dp)_x \big|_{N_x F_{p(x)}} \right)^{-1} (\vartheta_{p(x)}) & \forall x \notin K \\ 0 & \forall x \in K \end{cases}$$

(We won't need this explicit formula for Θ.)

Let $\Phi_t : S^3 \longrightarrow S^3$ be the flow of Θ. (called the monodromy flow)

Φ_t maps $F_{e^{2\pi i s}}$ to $F_{e^{2\pi i(s+t)}}$ homeomorphically

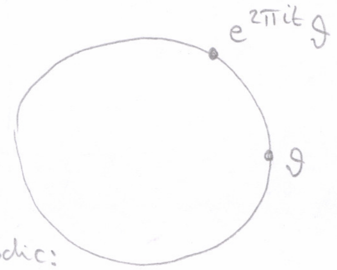
$\varphi := \Phi_1\big|_{F_1} : F_1 \longrightarrow F_1$ is the monodromy

<u>Example.</u> $K = T(a,b) = S^3 \cap \{y^a - x^b = 0\}$

Milnor showed that
$$p := \frac{y^a - x^b}{|y^a - x^b|} : S^3 \setminus K \longrightarrow S^1$$ is a fibre bundle.

$\zeta_t := e^{2\pi i t/ab}$

$$\Phi_t : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(x,y) \longmapsto (\zeta_t^a x, \zeta_t^b y)$$

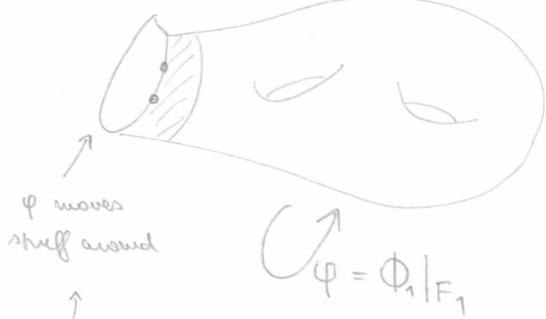


Observe that $p(\zeta_t^a x, \zeta_t^b y) = \frac{\zeta_t^{ab}}{|\zeta_t^{ab}|} p(x,y) = e^{2\pi i t} \underbrace{p(x,y)}_{\vartheta}$

$\varphi = \Phi_1\big|_{F_1}$ is the monodromy. We obtain that this is periodic:

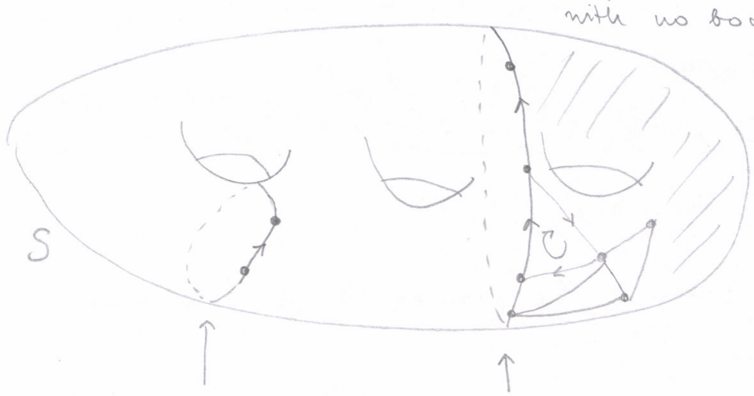
$$\Phi_1(x,y) = (\zeta_1^a x, \zeta_1^b y) = (e^{2\pi i/b} x, e^{2\pi i/a} y)$$

Thus the monodromy of a torus knot is periodic, up to an isotopy twisting the boundary.



This can be remedied with an isotopy.

Recall. $H_1(S;\mathbb{Z}) = \{\mathbb{Z}\text{-linear combinations of 1-dim oriented "pieces" with no boundary}\} / \{\text{bdries of 2-dim "pieces"}\}$



This is no boundary.

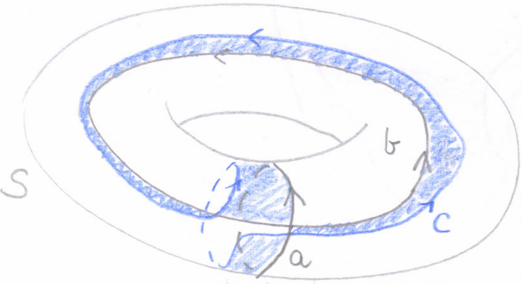
This is a boundary.

Here we don't specify what "pieces" mean; this holds for several homology theories.

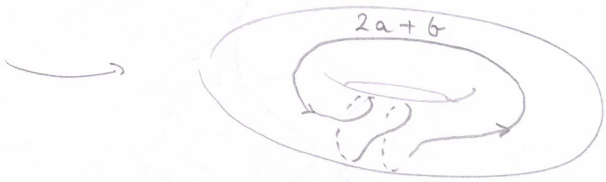
boundary

$1\cdot a + 1\cdot a' = 0 \in H_1(S;\mathbb{Z})$

$\Rightarrow -a$ is a with inverse orientation



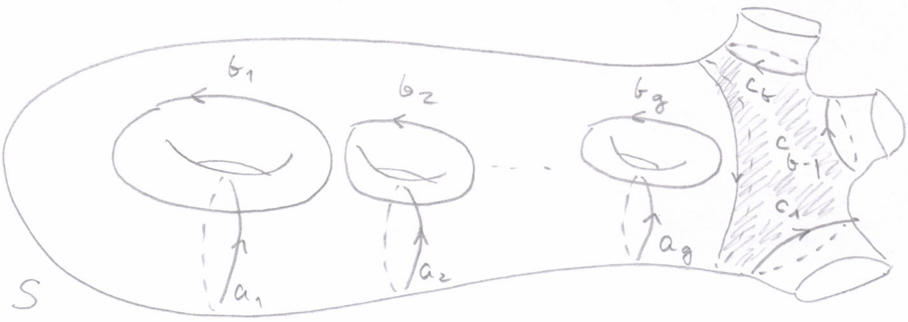
$1\cdot a + 1\cdot b = 1\cdot c$



Fact. S compact connected oriented surface $\Rightarrow H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^{2g(S)+b(S)-1}$ iso of $\mathbb{Z}$-modules where $g(S) =$ genus of S,

$b(S) = \#$ bdry components of $S \geq 1$

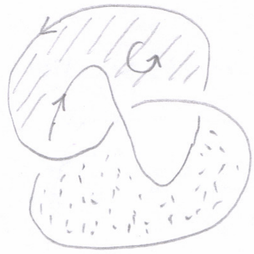
(If $b(S) = 0$ then $H_1(S,\mathbb{Z}) \simeq \mathbb{Z}^{2g(S)}$)



We take all but one bdry curves since the sum of all bdry curves is a bdry, hence 0.

$H_1(S;\mathbb{Z}) \simeq \langle a_1, \dots, b_1, \dots, c_1, \dots, c_{b-1} \rangle$

Seifert surfaces



Let $S \subseteq S^3$ be a Seifert surface.

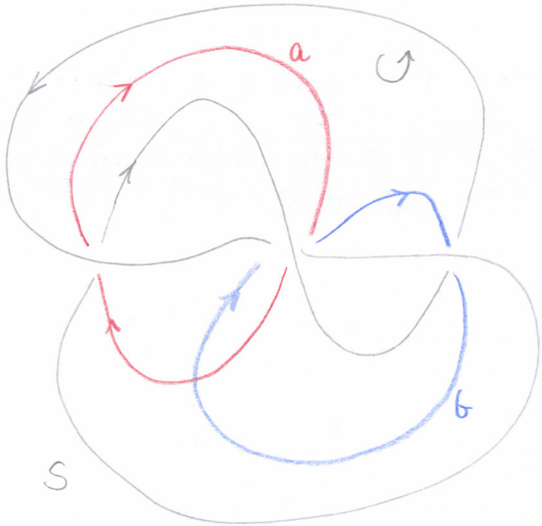
The Seifert form of S is the bilinear form $(\cdot,\cdot) : H_1(S;\mathbb{Z}) \times H_1(S;\mathbb{Z}) \longrightarrow \mathbb{Z}$

$$(a\ ,\ b) \longmapsto lk(a, b^+)$$

where b^+ is the curve obtained from b by pushing b off into the positive normal direction to S.

(Note that since $S \hookrightarrow S^3$ and oriented, every point of S has a normal vector.)

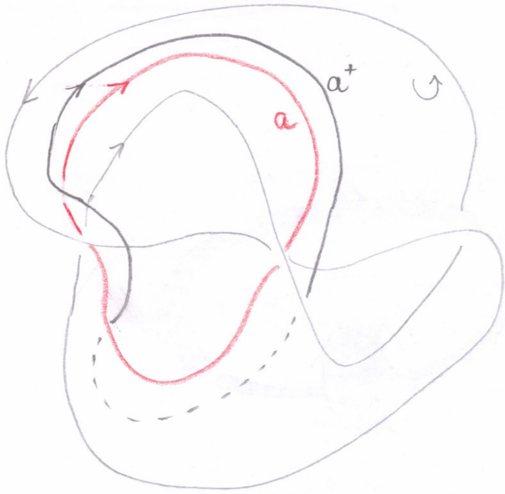
Ex.



$H_1(S;\mathbb{Z}) \simeq \langle a, b \rangle_{\mathbb{Z}} = \mathbb{Z}^2$

Seifert matrix:

$$\begin{array}{c c} & \begin{array}{cc} a^+ & b^+ \end{array} \\ \begin{array}{c} a \\ b \end{array} & \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix} \end{array}$$



right-hand rule:

$\Rightarrow$ positively linked

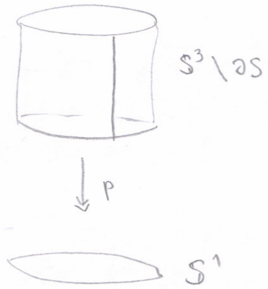
YouTube: ddreibel pb23

Let S be a fibre surface, $S^3 \setminus \partial S \twoheadrightarrow S^1$, $\partial S = K$

$\varphi : S \longrightarrow S$ monodromy map, $\varphi = \varphi_1$,

$\varphi_* : H_1(S;\mathbb{Z}) \longrightarrow H_1(S;\mathbb{Z})$ induced map

$[c] \longmapsto [\varphi(c)]$



φ_t is the monodromy flow

Lemma. $(v, w) = \big(\varphi_*(w), v\big) \quad \forall v, w \in H_1(S;\mathbb{Z})$

PROOF: $(v,w) = \mathrm{lk}(v, w^+) = \mathrm{lk}\left(v, \varphi_{1/2}(w)\right) = \mathrm{lk}\left(\varphi_{1/2}(v), \varphi_{1/2}\left(\varphi_{1/2}(w)\right)\right)$

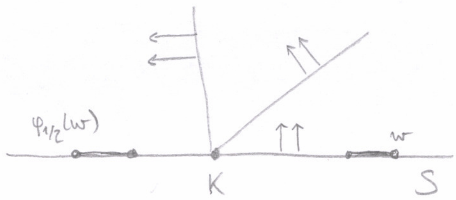
$\uparrow$ φ_t is an isotopy, flowing does not change lk

$\varphi_{1/2} \circ \varphi_{1/2} = \varphi_1$ (flow) $\Rightarrow$ $(v, w) = \mathrm{lk}\left(\varphi_{1/2}(v), \varphi_1(w)\right)$

$= \mathrm{lk}\left(\varphi_1(w), \varphi_{1/2}(v)\right)$ lk is symmetric

$= \mathrm{lk}\left(\varphi_1(w), v^+\right)$

$= \left(\varphi_1(w), v\right)$



example of why the monodromy map is not the identity:

it applies a Dehn twist.

the knot is fixed

example where monodromy map is Id: a disk

Choose a basis of $H_1(S,\mathbb{Z}) \cong \mathbb{Z}^n$. Write matrices A for $(\cdot,\cdot)$, M for φ_*,

$(x, y) = x^T A y$.

The Lemma says $x^T A y = (My)^T A x$.

$(My)^T A x = \underbrace{y^T M^T A x}_{\text{scalar}} = x^T A^T M y \quad \Rightarrow A = A^T M \quad \Rightarrow M = (A^T)^{-1} A$

$\rightarrow$ it is its own transpose

using that $\det A = \pm 1$ (believe this for now)

Def. Let S be a Seifert surface. $\boxed{\Delta_S(t)} := \det(tA^T - A)$ where A is the Seifert matrix. This is called the Alexander polynomial of S.

Thm. $\Delta_S(t) = \pm t^{\pm n} \Delta_{S'}(t)$ if $\partial S = \partial S' = K$, i.e. if S and S' bound the same surface.

Def. $\boxed{\Delta_K(t)} := \Delta_S(t)$ for some Seifert surface S of K.

Ex. $\left.\begin{array}{l} \Delta_{T(2,3)}(t) = t^2 - t + 1 \\ \\ \Delta_{\bigcirc}(t) = 1 \end{array}\right\} \Rightarrow$ the trefoil knot is not trivial.

Thm. Let S be a fibre surface with monodromy $\varphi : S \to S$. Then

$$\Delta_K(t) = \pm \chi_{\varphi_*}(t)$$

where $\chi_{\varphi_*}(t)$ denotes the characteristic polynomial of $\varphi_* : H_1(S;\mathbb{Z}) \to H_1(S;\mathbb{Z})$.

PROOF: $\Delta_K(t) = \det(tA^T - A)$ by def.

$$= \underbrace{\det(A^T)}_{\pm 1} \underbrace{\det(t \cdot \mathrm{Id} - \underbrace{(A^T)^{-1}A}_{M})}_{\chi_M(t) = \chi_{\varphi_*}(t)}$$

□

Rmk. $A - A^T$ is skew-symmetric; this turns out to be the intersection form for curves on S. It can be written as $\begin{pmatrix} \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & & \\ & \boxed{\begin{matrix} 0 & 1 \\ 1 & 0 \end{matrix}} & \\ & & \ddots \end{pmatrix} \to$ has det $= \pm 1$

But $A - A^T = A^T M - A^T$.

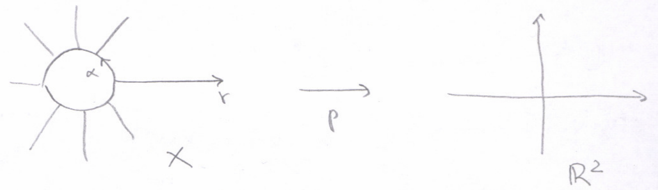
$\to \det A = \pm 1$

Cor. K fibred of genus $g \Rightarrow \Delta_K(t) = \pm 1 \cdot t^{2g} + \ldots$

Using this, one can show that certain knots are not fibred.

Polar coordinates

$$p : X := \mathbb{R}_{\geq 0} \times S^1 \longrightarrow \mathbb{R}^2$$
$$(r, \alpha) \longmapsto (r\cos\alpha, r\sin\alpha)$$

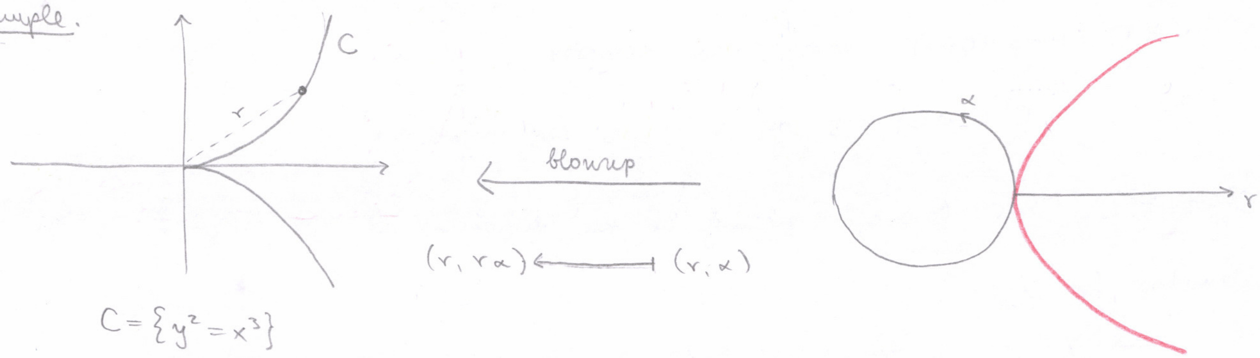


The only "bad" thing that happens here is that $p(\{0\}\times S^1) = \{(0,0)\}$, meaning that p is a diffeomorphism outside $(0,0) \in \mathbb{R}^2$.

$E := \{0\} \times S^1$ is called the exceptional divisor

$p|_{X\setminus E} : X \setminus E \longrightarrow \mathbb{R}^2$ diffeomorphism

Example.



$C = \{y^2 = x^3\}$

$t \mapsto (t^2, t^3)$ parametrisation

$r = \sqrt{t^4 + t^6} = t^2\sqrt{1+t^2} \approx t^2$ for $|t| \ll 1$

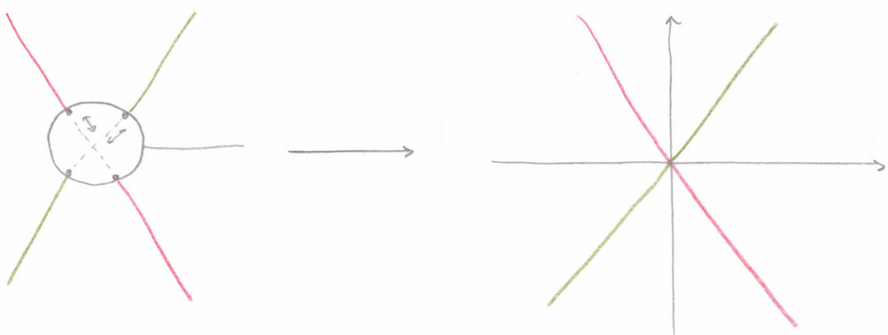
$(r, \alpha) \approx (t^2, t)$

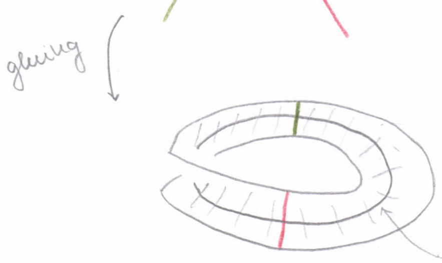
$\operatorname{tg} \alpha = \dfrac{t^3}{t^2} = t$

$\operatorname{tg} \alpha = \alpha + \frac{1}{3}\alpha^3 + \ldots \approx \alpha$ for $|\alpha| \ll 1$

This already exhibits the main property of blowups: they make singularities "less bad".

Example.





Möbius band

$$\mathbb{R}^2 \# \mathbb{P}^2\mathbb{R}$$
$$\uparrow$$
$$\mathbb{P}^1\mathbb{R} \cong S^1$$

We glue in the space of all directions to the origin.

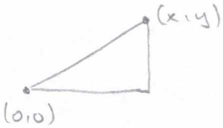
Complex blowup

$$\mathbb{C}^2 \longrightarrow \mathbb{C}^2$$
$$(u,v) \longmapsto (u, uv) = (x,y)$$

in local coordinates

$u = x$ is the radius

$v = \frac{y}{x}$ is the slope / angle



Globally: $Y := \left\{ ((x,y), [\alpha : \beta]) \in \mathbb{C}^2 \times \mathbb{P}^1\mathbb{C} \;\middle|\; \alpha y = \beta x \right\}$

think of this as the "algebraic form" of $\frac{y}{x} = \frac{\beta}{\alpha}$

$$\sigma : Y \longrightarrow \mathbb{C}^2$$
$$((x,y), (\alpha : \beta]) \longmapsto (x,y)$$

$E := \{0,0\} \times \mathbb{P}^1\mathbb{C} \longmapsto (0,0)$ exceptional divisor

$\sigma|_{Y \setminus E} : Y \setminus E \longrightarrow \mathbb{C}^2 \setminus \{0,0\}$ is an analytic isomorphism

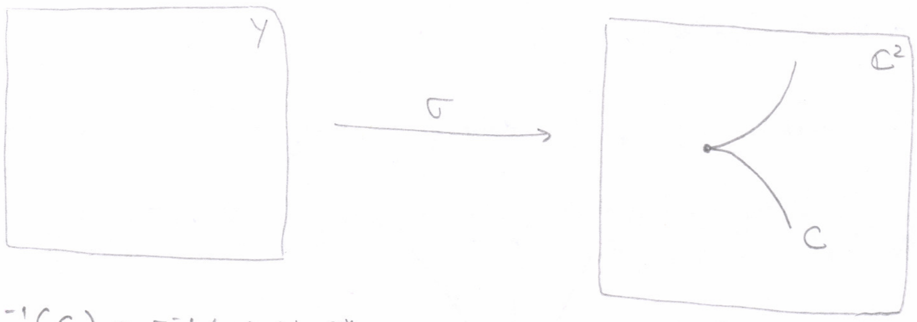
$Y \cong \mathbb{C}^2 \# \overline{\mathbb{P}^2\mathbb{C}}$ as before (gluing in the directions at the origin)

Local coordinates for Y:

$$\varphi_1 : \mathbb{C}^2 \longrightarrow Y$$
$$(u,v) \longmapsto ((u, uv), \underbrace{[1 : v]})$$

this is the sort of chart we like on $\mathbb{P}^1$

$$\varphi_2 : \mathbb{C}^2 \longrightarrow Y$$
$$(u,v) \longmapsto ((uv, v), \underbrace{[u : 1]})$$



$C = V(f)$

$\sigma^{-1}(C) = \underbrace{\sigma^{-1}(0)}_{E \cong \mathbb{P}^1\mathbb{C} \cong S^2} \amalg C^*$

φ_1

$\mathbb{C}^2$ $\qquad$ f $\qquad$ $\mathbb{C}^2$

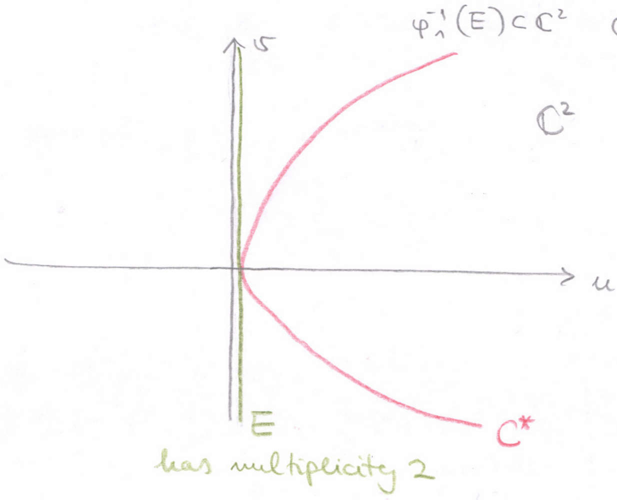
Let's look at $\sigma^{-1}(C) = (f\circ\sigma)^{-1}(0)$ in the chart φ_1:

$$\{(u,v)\in\mathbb{C}^2 \mid f(u,uv) = f\sigma\varphi_1(u,v) = 0\}$$

Example. $f = y^2 - x^3$ (as usual)

φ_1: $0 = f(u,uv) = u^2v^2 - u^3 = \underbrace{u^2}_{\varphi_1^{-1}(E)\subset\mathbb{C}^2}\underbrace{(v^2-u)}_{C^*}$ this describes the union of 2 curves

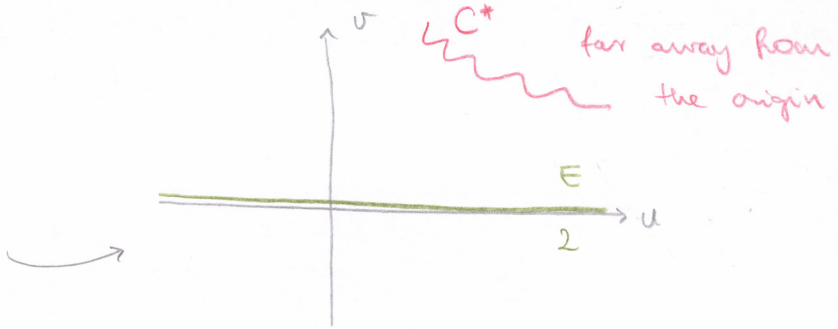
C^* strict transform of C



$\mathbb{C}^2 \longleftarrow$ We are in $\mathbb{C}^2$ again

$\Rightarrow$ we can blow up again!

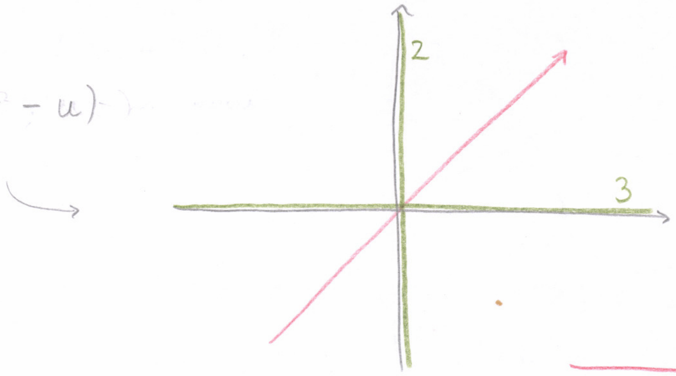
φ_2: $0 = f(uv, v) = v^2(1-u^3v)$



Now we blow up again:

$f_1(x,y) = x^2(y^2-x)$

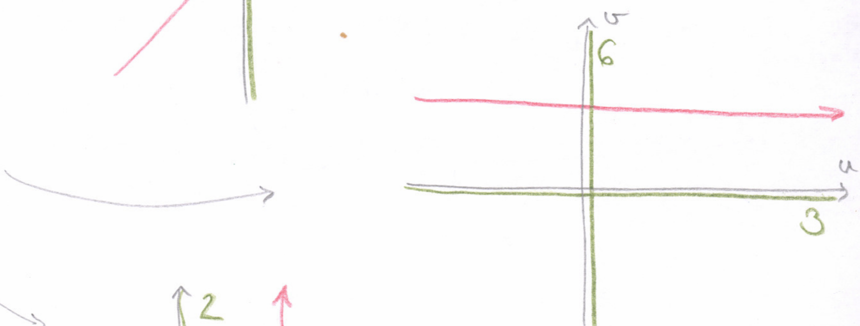
$f_1(uv,v) = u^2v^3(v-u)$



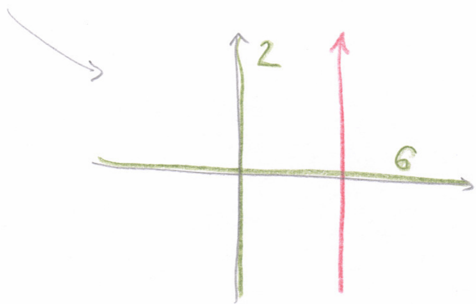
And again:

$f_2(x,y) = x^2y^3(y-x)$

$f_2(u,uv) = u^6v^3(v-1)$



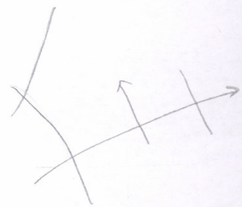
$f_2(uv,v) = u^2v^6(1-u)$



Summary:

If we were to go on:

Quantitatively nothing changes but the picture gets more complicated.

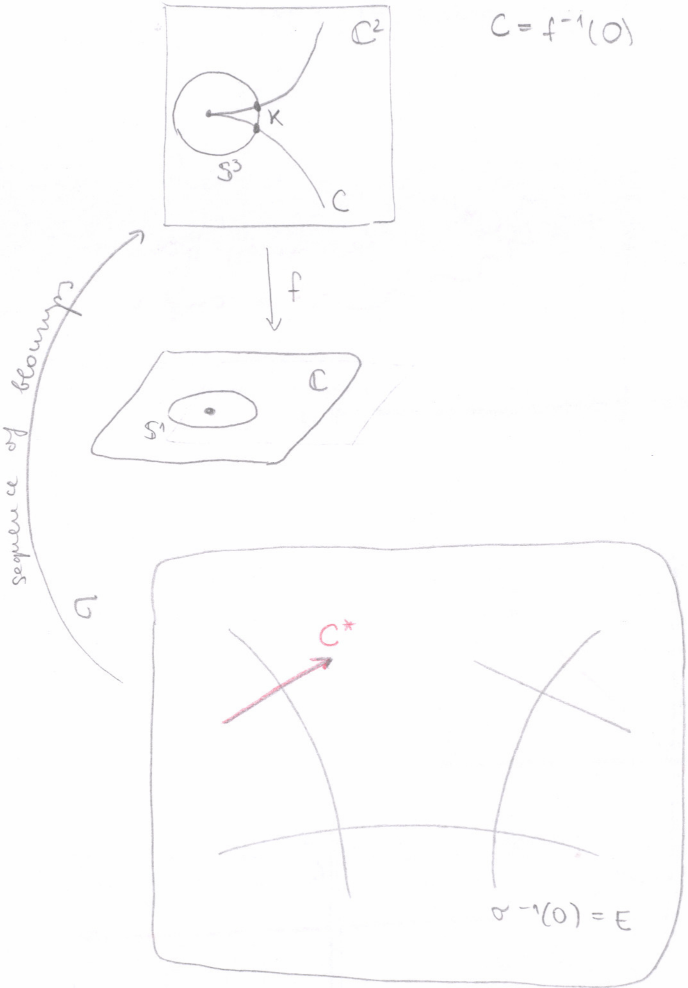


By resolving a singularity by a sequence of blowups, we mean arriving to stuff like we just did: every crossing is transversal, and everything is smooth.

Newton's algorithm is somehow analogous to this, and it works for the same reason Newton's algorithm does.

Resolution of singularities: [Hauser et al.] is a nice survey of this proof.
For fields of char 0: Hironaka. In positive char, this is still open.

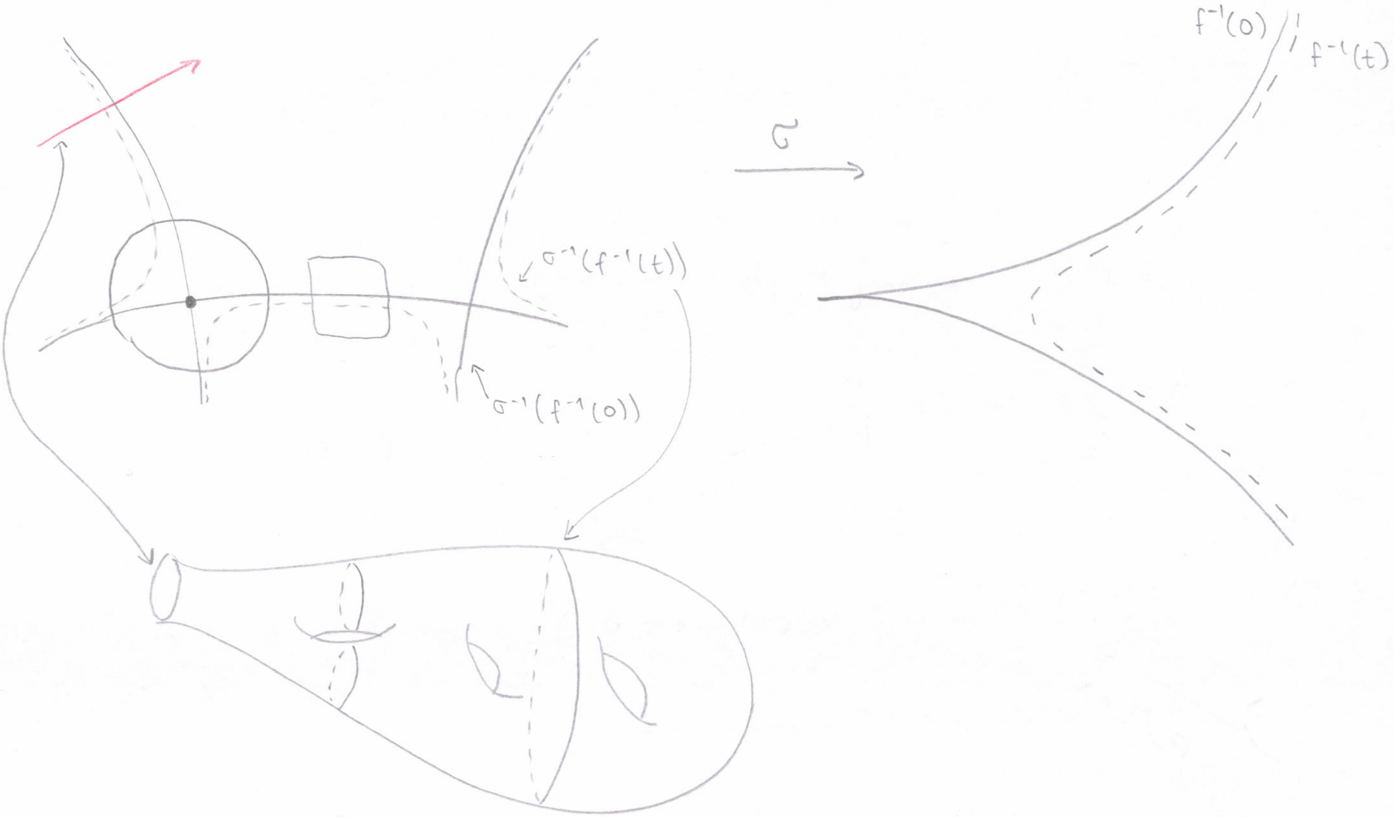
A'Campo 1975: understanding the monodromy of a singularity using blowups.



$C = f^{-1}(0)$

$$\begin{tikzcd} S^3 \setminus K \arrow[d, "\frac{f}{|f|}"', "\text{Milnor fibration}"] \\ S^1 \end{tikzcd}$$

If $0 \neq |t| \ll 1$ then $\sigma^{-1}(f^{-1}(t))$ is close to $\sigma^{-1}(f^{-1}(0))$ by continuity.

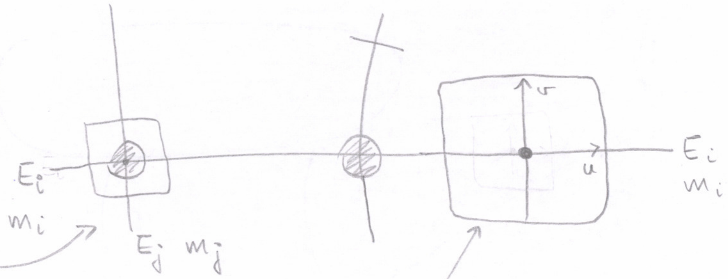


The blowup looks like this:

locally:

$0 = u^{m_j} v^{m_i} \cdot (1 - \ldots)$

$(t =)$

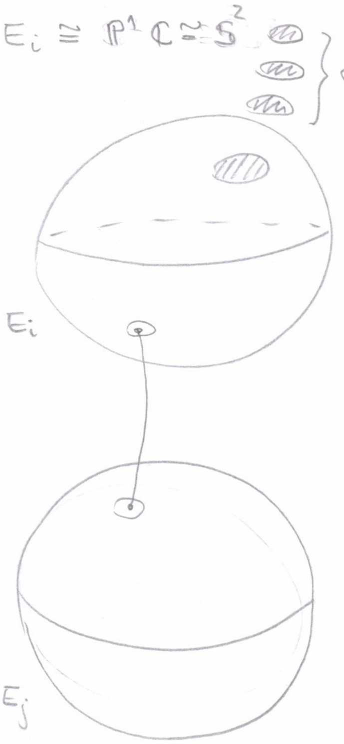


$v^{m_i}(\ldots) = 0$ $\quad E_i$

$v^{m_i} = t$ $\quad \sigma^{-1}(f^{-1}(t)) = F_t$

$E_i \cong \mathbb{P}^1\mathbb{C} \cong S^2$ $\left.\right\}$ $\sigma^{-1}(f^{-1}(t))$

m-fold covering



The monodromy is periodic outside the crossings

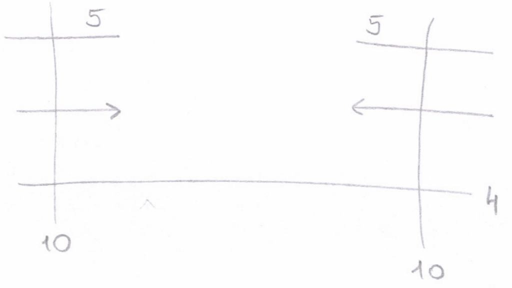
The fibre surface F of K decomposes into pieces (subsurfaces) $F = \bigcup_{i \in I} F_i$, one for each E_i.

F_i is an m_i-fold covering of $E_i \setminus \bigcup_{j \neq i} E_j$

The monodromy $\varphi|_{F_i}$ is the covering translation.

At each $E_i \cap E_j$ we get $\gcd(m_i, m_j)$ boundary components of F_i, F_j.

Example.



$E_i \cong \mathbb{P}^1\mathbb{C}$ Removing 2 pts:  $\cong$  cylinder

$E_i \setminus \bigcup_{i \neq j} E_j \cong \mathbb{P}^1\mathbb{C} \setminus \{3 \text{ pts}\} \cong$  pair of pants

$\chi(F_j) = 10\,\chi($  $) = -10 = 2 - 2g(F_j) - \#\pi_0(\partial F_j)$

Result: